Qualifying Examination, June 18, 2014
10:00 am–12:30pm, Room RH 114

Choose any 8 problems from 9

Table of your scores

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 1</td>
<td>/ 10</td>
</tr>
<tr>
<td>Problem 2</td>
<td>/ 10</td>
</tr>
<tr>
<td>Problem 3</td>
<td>/ 10</td>
</tr>
<tr>
<td>Problem 4</td>
<td>/ 10</td>
</tr>
<tr>
<td>Problem 5</td>
<td>/ 10</td>
</tr>
<tr>
<td>Problem 6</td>
<td>/ 10</td>
</tr>
<tr>
<td>Problem 7</td>
<td>/ 10</td>
</tr>
<tr>
<td>Problem 8</td>
<td>/ 10</td>
</tr>
<tr>
<td>Problem 9</td>
<td>/ 10</td>
</tr>
</tbody>
</table>

Total / 80
Notation: $D(z_0, r)$ denotes the open disc in $\mathbb{C}$ centered at $z_0$ and radius $r$.

1. Complete the following two problems.
   (a) Describe all entire holomorphic functions $f$ with $|f(z)| \leq |z|$ for all $z \in \mathbb{C}$;

   (b) Describe all entire holomorphic functions $f$ with $\lim_{z \to \infty} \frac{f(z)}{z} = 0$. 
2. Complete the following two problems.

(a) Evaluate \( \int_{|z|=1} \exp\left(\frac{1}{z^2}\right)dz \). (Here, \( \exp(z) = e^z \))

(b) Evaluate \( \int_0^\infty \frac{x^2}{1+x^7}dx \)
3. Let $f : D(0, 1) \to D(0, 1)$ be holomorphic with $f(0) = \frac{1}{3}$.
   (a) Give a sharp upper bound estimate for $|f'(0)|$.

   (b) Give an example of $f$ such that $|f'(0)|$ achieves the upper bound you obtained in Part (a).
4. Prove that there is an $N$ such that if $n \geq N$ then

$$\sum_{k=0}^{n} (k + 1)z^k \neq 0, \quad z \in D(0, 3/4)$$
5. Let $f_1, \ldots, f_n$ be holomorphic in a domain $D$ in $\mathbb{C}$ and $p \in (0, \infty)$. Prove
(a) $\sum_{j=1}^{n} |f_j(z)|^p$ is subharmonic in $D$
(b) if there is a $z_0 \in D$ such that $\sum_{j=1}^{n} |f_j(z_0)|^p \geq \sum_{j=1}^{n} |f_j(z)|^p$ for all $z \in D$, then $f_j$ is constant for $j = 1, 2, \cdots, n$. 
6. Let $D = \{ z \in \mathbb{C} : 1 < |z + 1| \text{ and } |z + 2| < 2 \}$. Construct a conformal holomorphic map which maps $D$ onto the unit disc $D(0, 1)$.
7. Let $D$ be a simply connected domain in $\mathbb{C}$ and $z_0 \in D$. If $\phi_1, \phi_2 \in \text{Aut}(D)$ such that

$$\phi_1(z_0) = \phi_2(z_0) \quad \text{and} \quad \phi'_1(z_0) = \phi'_2(z_0)$$

then $\phi_1 \equiv \phi_2$. (Hint: Try $D = D(0, 1)$ and $D = \mathbb{C}$ first)
8. Let \( f(z) \) be holomorphic in \( D =: D(0,1) \setminus \{0\} \) such that

\[
\int_D |f(z)| \, dA(z) < \infty
\]

Prove that \( z = 0 \) is either removable or a simple pole.
9. Let \( f_n : D(0, 1) \to D(0, 1) \setminus \{0\} \) be a sequence of holomorphic functions with \( \sum_{n=1}^{\infty} |f_n(0)|^2 < \infty \). Prove that
\[
\sum_{n=1}^{\infty} |f_n(z)|^3
\]
converges uniformly on \( D(0, 1/5) \).