ADVISORY/COMPREHENSIVE EXAM IN REAL ANALYSIS

Tuesday, June 14, 2016 — 9:00-11:30am

Problem	1	2	3	4	5	6	7	8	9	Σ
Points										

Each problem is worth 10 points. No books, notes, or calculators are allowed.

Student's name:

Problem 1.

Let $\{a_n\}$ is a sequence of nonnegative real numbers satisfying

$$a_{n+1} \leq a_n + \frac{(-1)^n}{n}, \qquad \forall n \in \mathbb{N}.$$

Prove that the sequence $\{a_n\}$ converges.

Problem 2.

Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be a continuous function. Assume that $f'(t)$ exists for all $t \in \mathbb{R}^1$. In general, $f'(t)$ does not have to be continuous. However, prove that the intermediate value property for f' holds, that is, the range of f' is connected.

Problem 3.

Let $f(t)$ be a real valued function that is continuous on $[0,1]$ and differentiable on $(0,1)$. Assume that $f(0) = 0$ and $|f'(t)| \leq |f(t)|$ for all $t \in (0,1)$. Prove that $f(t) \equiv 0$.

Problem 4.

Find, with justification, the value of the integral

$$\lim_{n\to\infty} \int_1^\infty \frac{n\sin(x^2/n)}{x^4}\,dx.$$

Problem 5.

Let $\{a_n\}$ be a sequence of real numbers. Prove that the sequence of functions $\{f_n\}$, $f_n : [0,1] \to \mathbb{R}$, defined by

$$f_n(x) = \int_0^x e^{(1+t^4)\sin^2(a_n t^5)} dt$$

has a subsequence that converges uniformly on $[0,1]$.

Problem 6.

a) Let $\{K_\alpha\}$ be a family of connected subsets of a metric space M such that any two sets from the family have non-empty intersection. Prove that the union $\cup_\alpha K_\alpha$ is connected.

b) Let $\{K_\alpha\}$ be a family of path connected subsets of a metric space M such that any two sets from the family have non-empty intersection. Is it true that the union $\cup_\alpha K_\alpha$ is path connected?

Problem 7.

Let M be a compact metric space, and $\epsilon > 0$. Show that there exists $n \in \mathbb{N}$ such that every set of n distinct points in M contains at least two points with distance between them less than ϵ.

Problem 8.

Let $U \subset \mathbb{R}^2$ be the region enclosed by the curve $\gamma(t) = (3t^2 - t^3, 6t - 2t^2)$, $t \in [0, 3]$. Find the area of U.

Problem 9.

Let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be a continuous function, and consider the function $F : \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$F(x, y) = \int_{D_{x,y}} f(u, v)dudv, \quad D_{x,y} = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 \leq x^2 + y^2\}.$$

Is $F(x, y)$ differentiable? If yes, find the differential DF.