1. Let $f \in L^p([0,1])$ for every $p \in [1, \infty]$. Show that
\[ ||f||_\infty = \lim_{p \to \infty} ||f||_p \]

2. Assume that $E$ is a subset of $\mathbb{R}^2$ and the distance between any two points in $E$ is a rational number. Show that $E$ is a countable set. (Note that this is actually true in any dimension.)

3. Let $f$ be a non-negative measurable function on $\mathbb{R}$. Prove that if $\sum_{n=-\infty}^{\infty} f(x + n)$ is integrable, then $f = 0$ a.e.

4. For each $n \geq 2$, we define $f_n : [0,1] \to \mathbb{R}$ as follows:
\[ f_n(x) = \begin{cases} n^2 & \text{for } x \in \left[ \frac{i}{n}, \frac{i}{n} + \frac{1}{n^2} \right] \text{ and } i = 0, \ldots, n - 1 \\ 0 & \text{otherwise} \end{cases} \]

(1) Show that
\[ \lim_{n \to +\infty} f_n = 0 \quad \text{for a.e. } x \in [0,1]. \]

(2) For $g \in C[0,1]$, what is
\[ \lim_{n \to +\infty} \int_{[0,1]} f_n(x)g(x) \, dx? \]
Justify your answer.

(3)(bonus problem): Is your answer to question (2) still valid for all $g \in L^\infty(0,1)$? Justify.

5. Let $A = \{(x,y) \in [0,1]^2 : x+y \notin \mathbb{Q}, xy \notin \mathbb{Q}\}$. Find $\int_A y^{-1/2} \sin x \, dm_2$ where $m_2$ is the Lebesgue measure in $\mathbb{R}^2$. Fully justify your steps.

6. Suppose that $f_n \in L^4(X,\mu)$ with $\|f_n\|_4 \leq 1$ and $f_n \to 0 \mu$-a.e. on $X$. Show that for every $g \in L^{4/3}(X,\mu)$,
\[ \int_X f_n(x)g(x)d\mu(x) \to 0 \]
as $n \to \infty$.
Hint: First consider the case when $X$ is of finite measure.