Instructions *Complete all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions. Problems are worth 10 points each. Bonus part of problem 4 is worth 3 extra points.*

1. Let $f \in L^p([0,1])$ for every $p \in [1,\infty]$. Show that
$$||f||_\infty = \lim_{p\to\infty} ||f||_p$$

2. Assume that E is a subset of $\mathbb{R}^2$ and the distance between any two points in E is a rational number. Show that E is a countable set. (Note that this is actually true in any dimension.)

3. Let f be a non-negative measurable function on $\mathbb{R}$. Prove that if $\sum_{n=-\infty}^{\infty} f(x+n)$ is integrable, then $f = 0$ a.e.

4. For each $n \geq 2$, we define $f_n : [0,1] \to \mathbb{R}$ as follows:
$$f_n(x) = \begin{cases} n^2 & \text{for } x \in [\frac{i}{n}, \frac{i}{n} + \frac{1}{n^3}] \text{ and } i = 0, 1, ..., n-1 \\ 0 & \text{otherwise} \end{cases}$$

   (1) Show that
$$\lim_{n\to+\infty} f_n = 0 \quad \text{for a.e. } x \in [0,1].$$

   (2) For $g \in C[0,1]$, what is
$$\lim_{n\to+\infty} \int_{[0,1]} f_n(x)g(x)\,dx?$$

   Justify your anwer.

   (3)(bonus problem): Is your answer to question (2) still valid for all $g \in L^\infty(0,1)$? Justify.

5. Let $A = \{(x,y) \in [0,1]^2 : x+y \notin \mathbb{Q}, xy \notin \mathbb{Q}\}$. Find $\int_A y^{-1/2} \sin x \, dm_2$ where m_2 is the Lebesgue measure in $\mathbb{R}^2$. Fully justify your steps.

6. Suppose that $f_n \in L^4(X,\mu)$ with $\|f_n\|_4 \leq 1$ and $f_n \to 0$ μ-a.e. on X. Show that for every $g \in L^{4/3}(X,\mu)$,
$$\int_X f_n(x)g(x)d\mu(x) \to 0$$
   as $n \to \infty$.
   Hint: First consider the case when X is of finite measure.