Preliminaries on groups

1 Group Theory

1.1 Basics, Lagrange, Cauchy

1. Let G be a group with an odd number of elements.

   (i) Prove that for each $a \in G$, the equation $x^2 = a$ has a unique solution.

   (ii) If G is abelian, prove that the product of all elements of G is 1_G.

2. Let G be a group of order $2m$ with m odd, and H a subgroup of G of order m. Prove that the product of all elements of G (in any order) is not an element of H.

3. Let a, b be elements in a group G such that $|a| = 25$ and $|b| = 49$. Prove that G contains an element of order 35.

4. Let a be en element of a group G such that $|a|$ is finite, and $H \trianglelefteq G$.

   (i) Prove that $|aH|$ divides $|a|$.

   (ii) If $|aH| = n$, prove that G contains an element of order n.

5. Let a, b be elements of a group G such that $a^5 = 1_G$ and $aba^{-1} = b^2$. Find $|b|$.

6. Suppose H is a subgroup of a finite group G, and H is the only subgroup of G with order n. Prove that H is normal.

7. Assume G is a group with order pq for some prime numbers p, q. Prove that G is abelian or $|Z(G)| = 1$.

8. Suppose that H and K are subgroups of a group G, and suppose that H and K have finite index in G. Show that the intersection $H \cap K$ also has finite index in G.

9. Let G be a group and H a subgroup of index $n < \infty$. Prove or disprove the following statements:

   (i) If $a \in G$, then $a^n \in H$.

   (ii) If $a \in G$, then for some $k, 1 \leq k \leq n$, we have $a^k \in H$.

10. Let G be a group. A subgroup H of G is called a characteristic subgroup of G if $\varphi(H) = H$ for every automorphism φ of G. Show that if H is a characteristic subgroup of N and $N \trianglelefteq G$, then $H \trianglelefteq G$. Is the same true if H is a only a normal subgroup of N?

11. Let G be a finite group and let M be a maximal subgroup of G (i.e., a proper subgroup that is not contained in another proper subgroup). Show that if $M \trianglelefteq G$, then $|G : M|$ is prime.

12. Let G be a finite group, H a subgroup of G and N a normal subgroup of G. Show that if $|H|$ is relatively prime to $|G : N|$, then $H \subseteq N$.

13. Let H be a subgroup of G. Show that the following are equivalent:

    (i) $x^{-1}y^{-1}xy \in H$ for all $x, y \in G$

    (ii) $H \trianglelefteq G$ and G/H is abelian.

14. If G is a finite group, must $S = \{g^2 | g \in G\}$ be a subgroup? Provide a proof or a counterexample.

15. Prove that every finite group of order at least 3 has a nontrivial automorphism.

16. Let G be a group of order $2n$, where n is odd, with a subgroup H of order n satisfying $xhx^{-1} = h^{-1}$ for all $h \in H$ and all $x \in G \setminus H$. Prove that H is commutative and that every element of $G \setminus H$ is of order 2.

17. Let G be a finite group. Let N be the subgroup of G generated by $\{x^2 | x \in G\}$. Prove that $N \trianglelefteq G$ and N contains $[G, G]$, the commutator subgroup of G.

18. Let $\varphi : G \to G'$ be a group homomorphism. Suppose $(|G|, |G'|) = 1$. Prove that φ is the trivial homorphism (i.e., $\varphi(x) = 1_{G'}$ for all $x \in G$).

1.2 One hour self test

Problem 1. How many distinct subgroups of index 5 are contained in $\mathbb{Z} \times \mathbb{Z}$?

Problem 2. Let G be a group and let N be a normal subgroup of index n. Show that $g^n \in N$ for all $g \in G$.

Problem 3. Suppose H is a subgroup of finite index of $\mathbb{C}^*$. Prove that $H = \mathbb{C}^*$. (HINT: in $\mathbb{C}^*$ you can always extract n-th root. Then consider the homomorphism $\mathbb{C}^* \to \mathbb{C}^*/H$ and apply Lagrange.)

Problem 4. Let $f : G \to H$ be a group homomorphism with H an abelian group. Suppose that N is a subgroup of G containing $\ker(f)$. Prove that N is a normal subgroup of G.