1 Preliminaries on Groups

1 Group Theory

1.1 Basics, Lagrange, Cauchy

1. Let $G$ be a group with an odd number of elements.
   (i) Prove that for each $a \in G$, the equation $x^2 = a$ has a unique solution.
   (ii) If $G$ is abelian, prove that the product of all elements of $G$ is $1_G$.

2. Let $G$ be a group of order $2m$ with $m$ odd, and $H$ a subgroup of $G$ of order $m$. Prove that the product of all elements of $G$ (in any order) is not an element of $H$.

3. Let $a, b$ be elements in a group $G$ such that $|a| = 25$ and $|b| = 49$. Prove that $G$ contains an element of order 35.

4. Let $a$ be an element of a group $G$ such that $|a|$ is finite, and $H \trianglelefteq G$.
   (i) Prove that $|aH|$ divides $|a|$.
   (ii) If $|aH| = n$, prove that $G$ contains an element of order $n$.

5. Let $a, b$ be elements of a group $G$ such that $a^5 = 1_G$ and $aba^{-1} = b^2$. Find $|b|$.

6. Suppose $H$ is a subgroup of a finite group $G$, and $H$ is the only subgroup of $G$ with order $n$. Prove that $H$ is normal.

7. Assume $G$ is a group with order $pq$ for some prime numbers $p, q$. Prove that $G$ is abelian or $|Z(G)| = 1$.

8. Suppose that $H$ and $K$ are subgroups of a group $G$, and suppose that $H$ and $K$ have finite index in $G$. Show that the intersection $H \cap K$ also has finite index in $G$.

9. Let $G$ be a group and $H$ a subgroup of index $n < \infty$. Prove or disprove the following statements:
   (i) If $a \in G$, then $a^n \in H$.
   (ii) If $a \in G$, then for some $k, 1 \leq k \leq n$, we have $a^k \in H$. 
10. Let $G$ be a group. A subgroup $H$ of $G$ is called a characteristic subgroup of $G$ if $\varphi(H) = H$ for every automorphism $\varphi$ of $G$. Show that if $H$ is a characteristic subgroup of $N$ and $N \trianglelefteq G$, then $H \trianglelefteq G$. Is the same true if $H$ is a only a normal subgroup of $N$?

11. Let $G$ be a finite group and let $M$ be a maximal subgroup of $G$ (i.e., a proper subgroup that is not contained in another proper subgroup). Show that if $M \trianglelefteq G$, then $|G : M|$ is prime.

12. Let $G$ be a finite group, $H$ a subgroup of $G$ and $N$ a normal subgroup of $G$. Show that if $|H|$ is relatively prime to $|G : N|$, then $H \subseteq N$.

13. Let $H$ be a subgroup of $G$. Show that the following are equivalent:
   (i) $x^{-1}y^{-1}xy \in H$ for all $x, y \in G$
   (ii) $H \trianglelefteq G$ and $G/H$ is abelian.

14. If $G$ is a finite group, must $S = \{g^2 | g \in G\}$ be a subgroup? Provide a proof or a counterexample.

15. Prove that every finite group of order at least 3 has a nontrivial automorphism.

16. Let $G$ be a group of order $2n$, where $n$ is odd, with a subgroup $H$ of order $n$ satisfying $xh x^{-1} = h^{-1}$ for all $h \in H$ and all $x \in G \setminus H$. Prove that $H$ is commutative and that every element of $G \setminus H$ is of order 2.

17. Let $G$ be a finite group. Let $N$ be the subgroup of $G$ generated by $\{x^2 | x \in G\}$. Prove that $N \trianglelefteq G$ and $N$ contains $[G, G]$, the commutator subgroup of $G$.

18. Let $\varphi : G \to G'$ be a group homomorphism. Suppose $\gcd(|G|, |G'|) = 1$. Prove that $\varphi$ is the trivial homomorphism (i.e., $\varphi(x) = 1_{G'}$ for all $x \in G$).

1.2 One hour self test

Problem 1. How many distinct subgroups of index 5 are contained in $\mathbb{Z} \times \mathbb{Z}$?
Problem 2. Let $G$ be a group and let $N$ be a normal subgroup of index $n$. Show that $g^n \in N$ for all $g \in G$.

Problem 3. Suppose $H$ is a subgroup of finite index of $\mathbb{C}^\ast$. Prove that $H = \mathbb{C}^\ast$. (HINT: in $\mathbb{C}^\ast$ you can always extract $n$-th root. Then consider the homomorphism $\mathbb{C}^\ast \to \mathbb{C}^\ast/H$ and apply Lagrange.)

Problem 4. Let $f : G \to H$ be a group homomorphism with $H$ an abelian group. Suppose that $N$ is a subgroup of $G$ containing $\ker(f)$. Prove that $N$ is a normal subgroup of $G$. 