Math 206 Practice Problems

1 Group Theory

1.1 Specific Groups \((S_n, A_n, D_n, \text{cyclic, abelian, } \ldots)\)

1. Show that \(\alpha\) and \(\alpha^{-1}\) have the same cycle type for all \(\alpha \in S_n\).

2. Let \(\alpha \in S_n\). Prove that \(\alpha\) and \(\alpha^2\) have the same cycle type \(\iff |\alpha|\) is odd.

3. Let \(\alpha \in S_n\). A fixed point of \(\alpha\) is a number \(m \in \{1, \ldots, n\}\) such that \(\alpha(m) = m\). Suppose that for all \(k \in \mathbb{N}\), either \(\alpha^k\) has no fixed point or \(\alpha^k = 1\). Prove that \(|\alpha|\) divides \(n\).

4. Let \(G\) be a subgroup of \(S_n\). Show that if \(G\) contains an odd permutation then \(G \cap A_n\) is of index 2 in \(G\).

5. Let \(G\) be a finite group with \(x, y\) distinct elements of order 2, and let \(H = \langle x, y \rangle\). Prove that \(H\) is a dihedral group.

6. (i) Find the maximal orders of elements of \(A_n\) and \(S_n\) for \(n = 7, 8\).
   (ii) Find a subgroup of \(A_6\) of order 8.

7. Show that any proper subgroup of \(Q_8\) is cyclic and normal.

8. How many homomorphisms are there from the group \(\mathbb{Z}/(2) \times \mathbb{Z}/(2)\) to \(S_3\)?

9. Let \(G = \mathbb{Z}/(81) \times \mathbb{Z}/(30) \times \mathbb{Z}/(16) \times \mathbb{Z}/(45)\).
   (i) What is the largest cyclic subgroup of \(G\)?
   (ii) How many elements of order 9 does \(G\) have?

10. Suppose \(G\) is a group that contains more than 12 elements of order 13. Prove that \(G\) is not cyclic.

11. Find the order of \((6, 4)\) in \(\mathbb{Z}/(24) \oplus \mathbb{Z}/(16)\).

12. Is there a surjective group homomorphism from \(\mathbb{Z}/(28)\) to \(\mathbb{Z}/(6)\)?
13. Show that if \( G \) is a finite cyclic group, then \( G \) has exactly one subgroup of order \( m \) for each positive integer \( m \) dividing \( |G| \).

14. Suppose \( n = pq \) with \( p, q \) distinct odd primes. Prove that \((\mathbb{Z}/n\mathbb{Z})^*\) is not cyclic.

15. List up to isomorphism the abelian groups \( A \) of order 108 satisfying both of the following:
   (i) \( A \) has an element of order 9;
   (ii) \( A \) does not have an element of order 24.

16. Let \( G \) be an abelian group. Let \( K = \{ a \in G : a^2 = 1 \} \) and let \( H = \{ x^2 : x \in G \} \). Show that \( G/K \cong H \).

17. Let \( N \trianglelefteq G \) such that every subgroup of \( N \) is normal in \( G \) and \( C_G(N) \subseteq N \). Prove that \( G/N \) is abelian.

18. Let \( G \) be an abelian group generated by \( x, y, z \) subject to the relations
   \[
   \begin{align*}
   15x + 3y &= 0 \\
   3x + 7y + 4z &= 0 \\
   18x + 14y + 8z &= 0
   \end{align*}
   \]
   (i) Write \( G \) as a product of two cyclic groups.
   (ii) How many elements of \( G \) have order 2?

### 1.2 Group actions, Sylow

1. Assume \( P \) is a Sylow \( p \)-subgroup of \( G \). Prove that \( P \) is the only Sylow \( p \)-subgroup of \( G \) contained in \( N_G(P) \).

2. Let \( N \) be a non-trivial normal subgroup of a finite \( p \)-group \( G \). Prove that \( N \) intersects \( Z(G) \) non-trivially.

3. Let \( G \) be a finite group of order \( n > 2 \). Let \( H \) be a subgroup of \( G \) such that \( r = [G : H] > 1 \). Assume that \( r! < 2n \). Prove that \( G \) is not a simple group. [Hint: construct a map from \( G \) into \( S_r \).]
4. Show that a group of order $2m$ with $m$ odd has a normal subgroup of order $m$. [Hint: construct a map $\varphi$ from $G$ into $S_{2m}$. Find an odd permutation in $\varphi(G)$.

5. Let $G$ be a finite group. Show that if $G$ has a normal subgroup $N$ of order 3 that is not contained in the center of $G$, then $G$ has a subgroup of index 2. [Hint: The group $G$ acts on $N$ by conjugation.]

6. Suppose a group $G$ has a subgroup $H$ with $|G : H| = n < \infty$. Prove that $G$ has a normal subgroup $N$ with $N \subseteq H$ and $|G : N| \leq n!$.

7. Let $G$ be a finite simple group containing an element of order 21. Show that every proper subgroup of $G$ has index at least 10.

8. Show that if $G$ is a group of order $392 = 2^3 \cdot 7^2$, then $G$ has a normal subgroup of order 7 or a normal subgroup of order 49.

9. Let $G$ be a group of order $pqr$, where $p > q > r$ are primes.
   (i) Show that if $p - 1$ is not divisible by $q$, then a Sylow $p$-subgroup of $G$ must be normal.
   (ii) Let $P$ be a Sylow $p$-subgroup of $G$ and assume $P$ is not normal in $G$. Show that a Sylow $q$-subgroup of $G$ must be normal.

10. Let $G$ be a group of order $p^3 - p$ where $p$ is a prime. Prove that the number of Sylow $p$-subgroups is either 1 or $p + 1$.

11. Let $G$ be a finite group and $p$ a prime. Show that the intersection of all Sylow $p$-subgroups of $G$ is a normal subgroup of $G$.

12. Let $G$ be a finite group and $p$ a prime. Let $N$ be a normal subgroup of $G$ and $H$ a Sylow $p$-subgroup of $G$. Show that
   (i) $HN/N$ is a Sylow $p$-subgroup of $G/N$, and
   (ii) $H \cap N$ is a Sylow $p$-subgroup of $N$.

13. Let $P$ be a Sylow $p$-subgroup of a group $G$ and let $K$ be a subgroup of $G$ containing $N_G(P)$. Show that $N_G(K) = K$.

14. Let $G$ be a group with exactly 3 elements of order 2. Prove that $G$ is not simple.
15. Assume $H$ is a nontrivial subgroup of $G$ such that $H \leq J$ for every non-trivial $J \leq G$. Prove that $H \leq Z(G)$.

16. Let $G$ be a finite simple group having a subgroup $H$ of prime index $p$. Prove that $p$ is the largest prime divisor of $|G|$.

17. Let $G$ be a group of order $2pq$, with $p, q$ odd primes (not necessarily distinct). Prove that $G$ is not simple.

18. Classify all groups of order $2012 = 2^2 \cdot 503$. [Hint: in one case, it may be helpful to know that the solutions of $x^4 \equiv 1 \pmod{503}$ are $\pm1$.]