Jumpstart 2012: test on groups

Problem 1. (i) How many elements of order 12 are there in S_7?
(ii) What is the maximal order of an element of S_7?
(iii) Give an example of two nonconjugate elements of S_7 that have the same order.

Problem 2. Find all normal subgroups of D_{2n}.

Problem 3. Show that if H is a cyclic normal subgroup of a finite group G, then every subgroup of H is a normal subgroup of G.

Problem 4. List all non-isomorphic abelian groups of order 54 that cannot be generated by 1 element but can be generated by 2 elements.

Problem 5. Show that if the center of a group G is of index n in G, then every conjugacy class of G has at most n elements. (HINT : consider conjugation action)

Problem 6. How many Sylow 2-subgroups does D_{2n} have, when n is odd?

Problem 7. Let H, K be two normal subgroups of G such that $H \cap K = \{e\}$ and any $g \in G$ can be written as a product $g = hk$ with $h \in H$ and $k \in K$. Show that G is isomorphic to the direct product group $H \times K$. (HINT: to show that $hk = kh$ use $(hkh^{-1})k^{-1} = h(kh^{-1}k^{-1})$ and normality)

Problem 8. Classify up to isomorphism all groups of order 45.