Summer Jump-Start Program for Analysis, 2012

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Based on my lecture note last year, part of the note was taken by Mis Anna Konstorum

1 Lecture 1: Numbers, Limits of Sequences (8/13/2012)

The main text for this short course is 'Principles of Mathematical Analysis' by W. Rudin.

1.1 Notations for sets of numbers:

(a) <u>Natural</u> numbers: $\mathbb{N} = \{1, 2, 3, ...\}$.

(b) <u>Integers</u>: $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$.

(c) <u>Rational</u>: $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\}$.

1.2 Three number fields:

a Rational number field: $\{\mathbb{Q}, +, \times\}$.

b Real number field: $\{\mathbb{R}, +, \times\}$.

c Complex number field: $\{\mathbb{C}, +, \times\}$.

1.3 Countable sets

- A set is said to be *countable* if $\exists$ a map $f : X \to \mathbb{N}$ which is 1-1 and onto (bijective).

  Problem 1: Prove that $\mathbb{Z}$ is a countable set.

Proof. We construct a map $f : \mathbb{Z} \to \mathbb{N}$ as follows:

$$f(n) = 2n, \text{ if } n > 0 \quad \text{and} \quad f(n) = (-2n + 1), \text{ if } n \leq 0.$$

It is easy to verify that $f : \mathbb{Z} \to \mathbb{N}$ is a one-to-one and onto map. Therefore, $\mathbb{Z}$ is countable.

Proposition 1.1. The following statements hold:

(a) If X, Y is countable, then $X \times Y = \{(x, y) : x \in X, y \in Y\}$ is also countable.

(b) If $X_1, X_2, ...X_n$ are countable, then $\cup_{j=1}^{\infty} X_j$ is countable.

(i.e.: a union of countable sets is countable).

(c) Every infinite subset of a countable set is countable.

(d) $\mathbb{Q}$ is countable.

Proof. We prove (a), and proofs for (b) and (c) are similar.

Since X and Y are countable, we can write $X = \{x_1, x_2, ..., x_n, ...\}$ and $Y = \{y_1, y_2, ..., y_n, ...\}$. Then $X \times Y$ form a matrix with entries (x_i, y_j). We can make an order as follows:

$$(x_1, y_1), (x_2, y_1), (x_1, y_2), (x_3, y_1), (x_2, y_2), (x_1, y_3), (x_4, y_1), (x_3, y_2), (x_2, y_3), (x_1, y_4), \cdots .$$

These form a map from $X \times Y$ to $\mathbb{N}$ which is bijective. So $X \times Y$ is countable. □

1.4 How to define $\mathbb{R}$?

In practical life, the rational numbers are not big enough for our usage. For example, the the diagonal of the unit square is not a rational number.

Example 1.1. Prove $\sqrt{2} \notin \mathbb{Q}$.

Proof. Assume $\sqrt{2} \in \mathbb{Q}$.

Then, $\sqrt{2} = p/q$, $(p, q) = 1$ (p, q, relatively prime) $\Rightarrow$

$\sqrt{2}q = p \ \Rightarrow 2q^2 = p^2 \ \Rightarrow 2|p^2 \Rightarrow 2|p \ \Rightarrow p = 2k.$

Then $2q^2 = 4k^2$ and $q^2 = 2k^2$, $2|q$.

We have shown that p, q have a common factor of 2, which contradicts our assumption that p, q are relatively prime. Therefore, $\sqrt{2} \notin \mathbb{Q}$. □

1.4.1 Limilt and Cauchy sequences

• Cauchy sequence: A sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ is said to be a Cauchy sequence if for any $\epsilon > 0$, there is N such that if $m, n \geq N$ then

$$|x_n - x_m| < \epsilon.$$

• Limit: We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of numbers has limit x as n tends to infinity,

$$\lim_{n\to\infty} x_n = x,$$

if for any $\epsilon > 0$, there is N such that if $n \geq N$ then

$$|x_n - x| < \epsilon.$$

1.4.2 Definition of $\mathbb{R}$

• Real number field $\mathbb{R}$ is the smallest number field containing $\mathbb{Q}$ such that every Cauchy sequence in $\mathbb{Q}$ has a limit in $\mathbb{R}$.

• Remark: $\mathbb{R}$=$\mathbb{Q} \cup \{$limits of Cauchy sequences in $\mathbb{Q}\}$.

1.5 Supremum or infimum

Definition 1.1. Boundedness

• A a subset X of $\mathbb{R}$ is *bounded above* if $\exists\ M \in \mathbb{R}$ such that $x \leq M$ for all $x \in X$.

Similarly, a subset X of $\mathbb{R}$ is *bounded below* if $\exists\ m \in \mathbb{R}$ such that $x \geq m$ for all $x \in X$.

Finally, X is *bounded* if $\exists\ m, M \in \mathbb{F}$ such that $m \leq x \leq M$ for all $x \in X$.

Definition 1.2. Supremum, infimum of subset in $\mathbb{R}$.

Let $X \subset \mathbb{R}$ be bounded above. The *supremum* of $X = \sup(X)$, is known as the the least upper bound of X. Which is defined as follows:

(i) $x \leq \sup(X)$ for all $x \in X$ ($\sup(X)$ is an upper bound of X)

(ii) For any $y \in \mathbb{R}$ and $y < \sup(X)$, then $\exists\ x_0 \in X$ such that $y < x_0$.

Similarly, the *infimum* of $X = \inf(X)$, is known as the the greatest lower bound of X which is defined such that

(i) $x \geq \inf(X)$ for all $x \in X$ ($\inf(X)$ is a lower bound of X)

(ii) For any $y \in \mathbb{R}$ and $y > \inf(X)$, then $\exists\ x_0 \in X$ such that $y > x_0$.

• Note: An ordered set $\mathbb{F}$, has the 'least upper bound property' if for any bounded above subset $X \subset \mathbb{F}$, $\sup(X)$ exists in $\mathbb{F}$.

Example 1.2. $\mathbb{Q}$ is an ordered field which does not satisfy the 'least upper bound property'.

Proof. Let $X = \{r \in \mathbb{Q}, r < \sqrt{2}\}$. It is not difficult to verify that $\sup(X) = \sqrt{2} \notin \mathbb{Q}$. Therefore, $\mathbb{Q}$ does not satisfy the least upper bound property.

Theorem 1.1. *The real number field $\mathbb{R}$ has the least upper bound property, with $\mathbb{Q} \subset \mathbb{R}$.*

Theorem 1.2. *Let $\{x_n\}_{n=1}^{\infty}$ be a monotonically increasing and bounded sequence in $\mathbb{R}$. Then, $\lim_{n\to\infty} x_n$ exists.*

Proof. Since $\{x_n\}_{n=1}^{\infty}$ is bounded above in $\mathbb{R}$, there exists $x = \sup\{x_n : n = 1, 2, ...\} \in \mathbb{R}$.
Now, we show that $\lim_{n\to\infty} x_n = x$ exists.
By definition of x, $x_n \leq x$ for all $n \in \mathbb{N}$. For any $\epsilon > 0$, since $(x - \epsilon)$ is not the upper bound of $\{x_n : n = 1, 2, ...\}$, by definition of x, there exists x_N such that $x_N > x - \epsilon$. Since we know that x_n is increasing when $n \geq N$, we get:

$$x + \epsilon > x \geq x_n \geq x_N > x - \epsilon, \quad \text{when } n \geq N.$$

Therefore, $|x - x_n| < \epsilon$ when $n \geq N$. So, $\lim_{n\to\infty} x_n = x$. $\square$

Example 1.3. $x_1 = 1$, $x_{n+1} = \sqrt{3 + x_n}$. Show that $\lim_{n\to\infty} x_n$ exists and find its value.

- *Strategy of the solving the problem*:

Step 1: We prove $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above, then $\lim_{n\to\infty} x_n = x$ exists in $\mathbb{R}$.

Step 2: We find x.

Proof. (a) We prove that x_n is increasing by induction:

(i) $x_1 = 1$, $x_2 = \sqrt{3+1} = 2$, so $x_2 > x_1$.

(ii) We assume $x_n > x_{n-1}$.

(iii) We show $x_{n+1} > x_n$. Since

$$x_{n+1} - x_n = \sqrt{3+x_n} - \sqrt{3+x_{n-1}} = \frac{x_n - x_{n-1}}{\sqrt{3+x_n} + \sqrt{3+x_{n-1}}} > 0.$$

Therefore, $x_{n+1} > x_n$, by mathematics induction, we have shown that $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence.

(b) We show, by induction again, that $\{x_n\}_{n=1}^{\infty}$ is bounded above:

(i) $x_1 = 1 < 3$, $x_2 = 2 < 3$

(ii) We assume $x_n \leq 3$.

(iii) Notice that: $x_{n+1} = \sqrt{3+x_n} \leq \sqrt{3+3} = \sqrt{6} < 3$.

Thus, by mathematical induction, $x_n \leq 3$, $n = 1, 2, \ldots.$

Therefore, since $\{x_n\}_{n=1}^{\infty}$ is bounded and monotonically increasing, $\lim_{n\to\infty} x_n = x \in \mathbb{R}$.

(c) Finally, we find the value of x:

Since $x_{n+1} = \sqrt{3+x_n}$, we take the limit of both sides:

$$x = \sqrt{3+x} \iff x^2 = 3 + x \iff x^2 - x - 3 = 0.$$

This implies $x = \dfrac{1 \pm \sqrt{13}}{2}$. Since $x \geq 1$, we have $x = \dfrac{1+\sqrt{13}}{2}$.

Theorem 1.3. $\lim_{n\to\infty}(1+\frac{1}{n})^{\frac{1}{n}}$ *exists, say e, which is called the natural number.*

Note: $(1+x)^{\alpha} \leq 1 + \alpha x$ for all $x > 0$ and $0 < \alpha < 1$. One can easily prove it.

Example 1.4.

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+n} = \lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1\dfrac{n^2+n}{n^2+1}}$$
$$= \left[\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1}\right]^1 = e$$

1.6 Basic techniques for computing limits.

(i) $\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$.

(ii) $\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$.

(iii) $\lim_{n\to\infty}(x_n \cdot y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$.

(iv) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$, where $\lim_{n\to\infty} y_n \neq 0$.

Theorem 1.4. *(Squeezed limit theorem): Assume*

$$x_n \le y_n \le z_n, \quad \textit{for } n \ge 1 \qquad \textit{and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x,$$

then $\lim_{n\to\infty} y_n$ *exists and equal* x*.*

Example 1.5. Let $\lim_{n\to\infty} x_n = x$. Prove

$$\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = x.$$

Proof. For any $\epsilon > 0$, since $\lim_{n\to\infty} x_n = x$, there is a N such that if $n \ge N$ then $|x_n - x| < \epsilon$.

Notice that

$$|\frac{x_1 + \cdots + x_n}{n} - x| = \left|\frac{x_1 - x + \cdots + x_{N-1} - x + x_N - x + \cdots + x_n - x}{n}\right| \le \frac{\sum_{j=1}^{N-1}|x_j| + N|x|}{n} + \epsilon.$$

Therefore, if $n \geq N_1 = \max\{N, 1 + (\sum_{j=1}^{N-1} |x_j| + N|x|)/\epsilon\}$, then

$$|\frac{x_1 + \cdots + x_n}{n} - x| < 2\epsilon.$$

Therefore, $\lim_{n\to\infty} \frac{x_1+\cdots+x_n}{n} = x$. $\square$

1.7 $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$

Definition 1.3.

(i) $\bar{x}_n = \sup\{x_m : m \geq n\}$.

Note that $\{\bar{x}_n\}_{n=1}^{\infty}$ is decreasing, and $\lim_{n\to\infty} \bar{x}_n$ exists, either finite or infinity.

(ii) $\underline{x}_n = \inf\{x_m : m \geq n\}$.

Note that $\{\underline{x}_n\}_{n=1}^{\infty}$ is increasing, and $\lim_{n\to\infty} \underline{x}_n$ exists, either finite or infinity.

(iii) $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \bar{x}_n$, $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \underline{x}_n$

Note that $\underline{x}_n \leq \bar{x}_n$, $\to \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

Theorem 1.5. $\lim_{n\to\infty} x_n$ *exists if and only if* $\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \sup x_n$

Exercise: prove the theorem!

Example 1.6. Let $x_n = (-1)^n$, $n = 1, 2, \ldots..$ Then

$$\overline{x}_n = \sup\{(-1)^m, m \geq n\} = 1, \quad \underline{x}_n = \inf\{(-1)^m, m \geq n\} = -1.$$

Therefore,$\limsup_{n\to\infty} x_n = 1 \neq -1 = \liminf_{n\to\infty} x_n$, $\lim_{n\to\infty}(-1)^n$ does not exist.

1.8 Exercises

1: Prove Theorem 1.1.

2: Let $\alpha > 0$ and $x_1 > \sqrt{\alpha}$, we define

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n})$$

Prove $\lim x_n = \sqrt{\alpha}$.

3: Let $\alpha > 1$ and let $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}$$

(a) Prove that x_{2n+1} is monotone decreasing.

(b) Prove that x_{2n} is monotone increasing.

(c) Prove that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

4: Find $\lim_{n\to\infty}(1 - \frac{1}{n})^n$.

5: Suppose that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

Prove

$$\lim_{n\to\infty} \frac{x_1 y_n + \cdots + x_n y_1}{n} = xy.$$