Summer Jump-Start Program for Analysis, 2012

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1 Lecture 2: Sequence, Series and power series (8/14/2012)

1.1 More on sequences

Example 1.1. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two bounded sequences. Show

$$\limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

Proof. Since

$$\begin{aligned}\overline{x_n + y_n} &= \sup\{x_m + y_m : m \ge n\}\\ &\le \sup\{x_m : m \ge n\} + \sup\{y_m : m \ge n\}\\ &= \overline{x_n} + \overline{y_n}\end{aligned}$$

we have $\lim_{n\to\infty} \overline{x_n + y_n} \le \lim_{n\to\infty}(\overline{x_n} + \overline{y_n}) = \lim_{n\to\infty} \overline{x_n} + \lim_{n\to\infty} \overline{y_n}$.

Therefore, $\limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$.

Definition 1.1. Given a sequence $\{x_n\}_{n=1}^{\infty}$. We say that $\{x_{n_j}\}_{j=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ if $n_1 < n_2 < .....$

Example 1.2. If we consider $x_n = (-1)^n$, $n = 1, 2, ...$, then

(i) $\{x_{2n}\}_{n=1}^{\infty}$ with $x_{2n} = 1$ is a subsequence of $\{x_n\}$.

(ii) $\{x_{2n+1}\}_{n=1}^{\infty}$ with $x_{2n+1} = -1$ is another subsequence.

Although $\lim_{n\to\infty} x_n$ does not exist, $\lim_{n\to\infty} x_{2n} = 1$, and $\lim_{n\to\infty} x_{2n+1} = -1$.

Theorem 1.1. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$, then $\lim_{n\to\infty} x_n = x \iff$ for any subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, one has $\lim_{k\to\infty} x_{n_k} = x$.*

Proof. It is straight forward.

Proposition 1.1. Given bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ s.t. $\limsup_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k}$.

Proof. The idea for the proof:

$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \overline{x}_n$ where $\overline{x}_n = \sup\{x_n : m \geq n\}$.

Then $\bar{x}_k - 1/k$ is not an upper bound, one can choose n_k s.t. $\bar{x}_k \geq x_{n_k} > \bar{x}_k - 1/k$, inductively, we can choose $n_1 < n_2 < \cdots$. Then $\lim_{k\to\infty} x_{n_k} = \limsup x_n$. □

Theorem 1.2. *Bolzanno-Weierstrass Theorem:*

Every bounded sequence in $\mathbb{R}$ *has a convergent subsequence*

Proof. We consider two cases:

(1) $\{x_n\}_{n=1}^{\infty}$ is finite: there must be 1 repeat ∞ many times. Then, there is a subsequence $\{x_{n_k}\} = x$.

(2) $\{x_n\}_{n=1}^{\infty}$ is infinite. Then, we choose m, M such that $\{x_n\}_{n=1}^{\infty} \subset I_0 := [m, M]$, we choose a half of the interval from I_k and called I_{k+1} with I_{k+1} contains infinite set of $\{x_n\}_{n=1}^{\infty}$. Choose $\{x_{n_k}\}_{k=1}^{\infty}$ with $x_{n_k} \in I_k$. Then $|x_{n_k} - x_{n_\ell}| \leq \dfrac{2M}{2^k}$, $\ell > k$. So, $\{x_{n_k}\}$ is a Cauchy sequence, which has a limit. □

Theorem 1.3. $\mathbb{Q}$ *is dense in* $\mathbb{R}$

Because of the following theorem on the structure of real numbers:

Theorem 1.4. *Every* $x \in \mathbb{R}$ *can be represented as* $x = p + \sum_{j=1}^{\infty} \frac{a_j}{10^j}, a_j \in \{0, 1, ..., q\}, p \in \mathbb{Z}$. *So, we then get* $1 = 0.9999.... = 1.0$.

1.2 Definition and basic test for a convergent series

Definition 1.2. We say that $\sum_{n=1}^{\infty} a_n$ converges (to a finite number $\in \mathbb{R}$), if $\lim_{n\to\infty} S_n$ exists, where $S_n = \sum_{k=1}^{\infty} a_k$.

- **Question:** How to test $\sum_{n=1}^{\infty} a_n$ converges or diverges ?

Proposition 1.2. (Necessary condition) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

Proof. Notice that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} S_{n+1} - S_n = S - S = 0$. The proof is complete. $\square$

Example 1.3. Prove $\sum_{n=1}^{\infty} \dfrac{(-1)^n + n}{n + 100}$ diverges.

Proof. Since $\lim_{n\to\infty} \dfrac{(-1)^n + n}{n + 100} = 1 \neq 0$, we have that $\sum_{n=1}^{\infty} \dfrac{(-1)^n + n}{n + 100}$ diverges. $\square$

Definition 1.3. We say that $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.

Notice the $\sum_{n=1}^{\infty} a_n \Leftrightarrow \{S_n\}_{n=1}^{\infty}$ converges. One easily has

Theorem 1.5. *(Cauchy criteria) The series $\sum_{n=1}^{\infty} a_n$ converges if and only if for $\epsilon > 0$, there is an N such that if $m > n \geq N$ then $|\sum_{k=n+1}^{m} a_k| < \epsilon$.*

Corollary 1.6. *$\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges. The converse is not true*

Note: First part is from: Cauchy test and $|\sum_{k=n}^{m} a_k| \leq \sum_{k=n+1}^{m} |a_k|$. The second part is from

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges, but it does not converges absolutely.

1.3 Series with non-negative general terms: $a_n \geq 0$

Theorem 1.7. *(Comparison test) Assume $0 \leq b_n \leq a_n \leq c_n$, $n = 1, 2, ...,$ then*

(i) If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} b_n$ diverges $(+\infty)$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 1.8. *p-series*
$\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ *converges if $p > 1$ and diverges to $+\infty$ if $p \leq 1$.*

Proof. If $p > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq 1 + \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^p} dx = 1 + \int_1^{\infty} \frac{1}{x^p} dx = 1 + \frac{1}{p-1} < \infty$$

If $p \leq 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \geq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^p} dx = 1 + \int_1^{\infty} \frac{1}{x^p} dx = \infty.$$

□

Example 1.4. Determine if $\sum_{n=1}^{\infty} \dfrac{(-1)^n + n^2}{n^4 + 10n + 1}$ converges.

Solution: Since

$$0 \leq \frac{(-1)^n + n^2}{n^4 + 10n + 1} \leq \frac{2n^2}{n^4} \leq \frac{2}{n^2}.$$

Since $\sum_{n=1}^{\infty} \dfrac{1}{n^2} < +\infty$ *(it is a p-series, with* $p = 2$*), then by the comparison test, the series converges.*

Theorem 1.9. *(Integral test) Assume that* $f(x)$ *is a decreasing non-negative function on* $[1, \infty)$*.*

(i) $0 \leq a_n \leq f(n)$ *and* $\int_1^{\infty} f(x)dx < +\infty$*, then* $\sum_{n=1}^{\infty} a_n$ *converges;*

(ii) $a_n \geq f(n)$ *and* $\int_1^{\infty} f(x)dx = +\infty$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. (i) Since $a_{n+1} \leq f(n+1) \leq \int_n^{n+1} f(x)dx$, one has $\sum_{n=1}^{\infty} a_n \leq a_1 + \sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx = a_1 + \int_1^{\infty} f(x)dx < +\infty$. Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Since $a_n \geq f(n)$, one has $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx = \int_1^{\infty} f(x)dx = +\infty$. Therefore, $\sum_{n=1}^{\infty} a_n$ diverges.

□

Example 1.5. $\sum_{n=1}^{\infty} \dfrac{1}{n(\ln(n+1))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution. Let $f_p(x) = \frac{1}{x(\log(x+1))^p}$*. If* $p \neq 1$*, then*

$$\int_1^{\infty} f(x)dx = \frac{1}{-p+1} \ln(x+1))^{p-1}\Big|_{x=1}^{\infty}$$

This is finite when $p > 1$, infinity when $p \leq 1$. Moreover, $a_n = f(n)$. The result follows from integral test.

Theorem 1.10. *Ratio test for series with general term positive.*

Let $a_n > 0$, $n = 1, 2, ..$ and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$.

(i) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges;

(ii) If $r > 1$, hen $\sum_{n=1}^{\infty} a_n$ diverges;

(iii) When $r = 1$, the test fails.

Example 1.6. Determine if $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$ converges.

Solution: $a_n = \dfrac{2^n}{n!} > 0$, $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{(n+1)!}\dfrac{n!}{2^n} = \dfrac{2}{n+1} \to 0$ as $n \to \infty$.

So, the series converges.

Example 1.7. Let $a_n = 1/n$ and $b_n = 1/n^2$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{b_{n+1}}{b_n} = 1.$$

But, $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ converges.

Theorem 1.11. *Root test*

If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = r$, *then:*

1. *if $r < 1$, then $\sum_{n=1}^{\infty} |a_n|$ converges.*

2. *if $r > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.*

3. *$r = 1$, test fails.*

Example 1.8. Find interval x such that $\sum_{n=1}^{\infty} \dfrac{n^2}{8^n} x^{3n}$ converges.

Solution. Let $a_n = \dfrac{n^2}{8^n} x^{3n}$. Then

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{n^2}{8^n}|x|^3} = \frac{|x|^3}{8}\sqrt[n]{n^2} \to \frac{|x|^3}{8} < 1, \qquad \textit{when } |x| < 2.$$

Therefore, the series converges for all $x \in (-2, 2)$.

• *We have discussed how to test if a series with non-negative terms converges. In general, testing the convergence of $\sum_{n=1}^{\infty} a_n$ with general a_n is very difficult. The following identity is a very useful tool.*

Theorem 1.12. *Abel's Identity*

$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} s_k(b_k - b_{k+1}) + s_q b_q - s_{p-1} b_p$

where $s_k = \sum_{l=1}^{k} a_l$, $s_0 = 1$.

Proof. Notice that

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} (s_k - s_{k-1}) b_k = \sum_{k=p}^{q-1} (s_k b_k - s_k b_{b+1}) + s_q b_q - s_{p-1} b_p = \sum_{k=p}^{q} s_k(b_k - b_{k+1}) + s_q b_q - s_{p-1} b_p.$$

The proof is complete. □

Theorem 1.13. *Test for alternating series*

Let $\{b_k\}_{k=1}^{\infty}$ is a decreasing sequence and $\lim_{n\to\infty} b_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Proof. Apply Abel's identity with $a_n = (-1)^n$, $s_n = 0$ if n is even, -1 if n is odd.

Then the above theorem follows directly from the following more general theorem □

Theorem 1.14. *Let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be two sequences of numbers such that*

(i) $\{s_n = \sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ is bounded;

(ii) $b_n \geq b_{n+1}$ for $n \geq 1$ and $b_n \to 0$ as $n \to \infty$.

Then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof. For any $\epsilon > 0$, need to find N such that if $m, n > N$ then $|\sum_{k=n}^{m} a_k b_k| < \epsilon$. Since $\{s_n\}_{n=1}^{\infty}$

is bounded, there is $M > 0$ such that $|s_n| \leq M$ for all $n \geq 1$, we have

$$\begin{aligned}\left|\sum_{k=n}^{m} a_k b_k\right| &= \left|\sum_{k=n}^{m} s_k(b_k - b_{k+1}) + s_m b_m - s_{n-1} b_n\right| \\ &\leq \sum_{k=n}^{m} |s_k|(b_k - b_{k+1}) + (|s_m| + |s_n|)(|b_m| + |b_n|) \\ &\leq M(b_n - b_{m+1}) + 2M(|b_n| + |b_m|) \\ &\leq 3M(|b_n| + |b_m|).\end{aligned}$$

Since $\lim_{n\to\infty} b_n = 0$, there exists N such that if $n \geq N$, we have $|b_n| \leq \epsilon/6M$. Therefore, when $m > n \geq N$, we have $|\sum_{k=n}^{m} a_k b_k| \leq 6M \cdot \epsilon/6M = \epsilon$. Therefore, $\sum_{k=1}^{\infty} a_k b_k$ converges by the Cauchy criteria of convergence. □

Example 1.9. Determine if $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\ln(n+1)}$ converges.

Solution. *Since* $\ln(n+2) \geq \ln(n+1)$, $n = 1, 2, ...,$ *then* $\dfrac{1}{\ln(n+2)} \leq \dfrac{1}{\ln(n+1)}$, $n = 1, 2, ...$ *and* $\lim_{n\to\infty} \dfrac{1}{\ln(n+1)} = 0$. *By the alternating series test,* $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\ln(n+1)}$ *converges.*

Exercise: *Show that* $\sum_{n-1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$ *converges.*

1.4 Exercise

1. *Prove the following series converges.*
   - *(a)* $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
   - *(b)* $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$
2. *Prove* $\mathbb{Q} \times \mathbb{Q}$ *is dense in* $\mathbb{R}^2$.
3. *Investigate the behavior (convergence and divergence) of* $\sum_{n=1}^{\infty} a_n$ *if*

   (a) $a_n = \sqrt{n+1} - \sqrt{n}$; *(b)* $a_n = (\sqrt{n+1} - \sqrt{n})/n$; *(c)* $a_n = (n^{1/n} - 1)^n$ *and*

   (d) $a_n = \frac{1}{1+z^n}$, *where* z *is complex number.*
4. *If* $a_n \geq 0$ *and* $\sum_{n=1}^{\infty} a_n$ *converges then* $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ *converges.*
5. *If* $\sum_{n=1}^{\infty} a_n$ *converges and in* $\{b_n\}$ *is monotonic and bounded, prove that* $\sum_{n=1}^{\infty} a_n b_n$ *converges.*
6. *Suppose that* $a_n > 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$.

(a) Prove that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ *diverges;*

(b) What can be said about

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}, \quad \sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n},$$

(c) Let $s_n = \sum_{k=1}^{n} a_k$*. Prove*

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{s_n^2}$ *converges.*

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2.1 More on sequences

Example 2.1. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two bounded sequences. Show

$$\lim\sup_{n\to\infty}(x_n + y_n) \leq \lim\sup_{n\to\infty} x_n + \lim\sup_{n\to\infty} y_n.$$

Proof. Since

$$\begin{aligned}\overline{x_n + y_n} &= \sup\{x_m + y_m : m \geq n\} \\ &\leq \sup\{x_m : m \geq n\} + \sup\{y_m : m \geq n\} \\ &= \overline{x_n} + \overline{y_n}\end{aligned}$$

we have $\lim_{n\to\infty} \overline{x_n + y_n} \leq \lim_{n\to\infty}(\overline{x_n} + \overline{y_n}) = \lim_{n\to\infty} \overline{x_n} + \lim_{n\to\infty} \overline{y_n}$.

Therefore, $\lim\sup_{n\to\infty}(x_n + y_n) \leq \lim\sup_{n\to\infty} x_n + \lim\sup_{n\to\infty} y_n$.

Definition 2.1. Given a sequence $\{x_n\}_{n=1}^{\infty}$. We say that $\{x_{n_j}\}_{j=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ if $n_1 < n_2 < .....$

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Although $\lim_{n\to\infty} x_n$ does not exist, $\lim_{n\to\infty} x_{2n} = 1$, and $\lim_{n\to\infty} x_{2n+1} = -1$.

Theorem 2.1. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$, then $\lim_{n\to\infty} x_n = x \iff$ for any subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, one has $\lim_{k\to\infty} x_{n_k} = x$.*

Proof. It is straight forward.

Proposition 2.1. Given bounded sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ s.t. $\limsup_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k}$.

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Then $\bar{x}_k - 1/k$ is not an upper bound, one can choose n_k s.t. $\bar{x}_k \geq x_{n_k} > \bar{x}_k - 1/k$, inductively, we can choose $n_1 < n_2 < \cdots$. Then $\lim_{k\to\infty} x_{n_k} = \limsup x_n$. □

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Every bounded sequence in $\mathbb{R}$ has a convergent subsequence

Proof. We consider two cases:

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Theorem 2.3. *$\mathbb{Q}$ is dense in $\mathbb{R}$*

Because of the following theorem on the structure of real numbers:

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- **Question:** How to test $\sum_{n=1}^{\infty} a_n$ converges or diverges ?

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Proof. Notice that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} S_{n+1} - S_n = S - S = 0$. The proof is complete. □

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Proof. Since $\lim_{n\to\infty} \dfrac{(-1)^n + n}{n+100} = 1 \neq 0$, we have that $\sum_{n=1}^{\infty} \dfrac{(-1)^n + n}{n+100}$ diverges. □

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Corollary 2.6. *$\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges. The converse is not true*

Note: First part is from: Cauchy test and $|\sum_{k=n}^{m} a_k| \leq \sum_{k=n+1}^{m} |a_k|$. The second part is from

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges, but it does not converges absolutely.

2.3 Series with non-negative general terms: $a_n \geq 0$

Theorem 2.7. *(Comparison test) Assume* $0 \leq b_n \leq a_n \leq c_n$, $n = 1, 2, ...,$ *then*

(i) If $\sum_{n=1}^{\infty} c_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(ii) If $\sum_{n=1}^{\infty} b_n$ *diverges* $(+\infty)$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Theorem 2.8. *p-series*
$\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges to* $+\infty$ *if* $p \leq 1$.

Proof. If $p > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq 1 + \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^p} dx = 1 + \int_1^{\infty} \frac{1}{x^p} dx = 1 + \frac{1}{p-1} < \infty$$

If $p \leq 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \geq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^p} dx = 1 + \int_1^{\infty} \frac{1}{x^p} dx = \infty.$$

□

Example 2.4. Determine if $\sum_{n=1}^{\infty} \frac{(-1)^n + n^2}{n^4 + 10n + 1}$ converges.

Solution: Since

$$0 \leq \frac{(-1)^n + n^2}{n^4 + 10n + 1} \leq \frac{2n^2}{n^4} \leq \frac{2}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$ (it is a p-series, with $p = 2$), then by the comparison test, the series converges.

Theorem 2.9. *(Integral test) Assume that* $f(x)$ *is a decreasing non-negative function on* $[1, \infty)$.

(i) $0 \leq a_n \leq f(n)$ *and* $\int_1^{\infty} f(x)dx < +\infty$*, then* $\sum_{n=1}^{\infty} a_n$ *converges;*

(ii) $a_n \geq f(n)$ *and* $\int_1^{\infty} f(x)dx = +\infty$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. (i) Since $a_{n+1} \leq f(n+1) \leq \int_n^{n+1} f(x)dx$, one has $\sum_{n=1}^{\infty} a_n \leq a_1 + \sum_{n=1}^{\infty} \int_n^{n+1} f(x)dx = a_1 + \int_1^{\infty} f(x)dx < +\infty$. Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

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□

Example 2.5. $\sum_{n=1}^{\infty} \dfrac{1}{n(\ln(n+1))^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution. Let $f_p(x) = \frac{1}{x(\log(x+1))^p}$. If $p \neq 1$, then

$$\int_1^{\infty} f(x)dx = \frac{1}{-p+1} \ln(x+1))^{p-1}\Big|_{x=1}^{\infty}$$

This is finite when $p > 1$, infinity when $p \leq 1$. Moreover, $a_n = f(n)$. The result follows from integral test.

Theorem 2.10. *Ratio test for series with general term positive.*

Let $a_n > 0$, $n = 1, 2, ..$ *and* $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = r$.

(i) If $r < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges;*

(ii) If $r > 1$, *hen* $\sum_{n=1}^{\infty} a_n$ *diverges;*

(iii) When $r = 1$, *the test fails.*

Example 2.6. Determine if $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$ converges.

Solution: $a_n = \dfrac{2^n}{n!} > 0$, $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}}{(n+1)!} \dfrac{n!}{2^n} = \dfrac{2}{n+1} \to 0$ as $n \to \infty$.

So, the series converges.

Example 2.7. Let $a_n = 1/n$ and $b_n = 1/n^2$. Then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{b_{n+1}}{b_n} = 1.$$

But, $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ converges.

Theorem 2.11. *Root test*

If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = r$, *then:*

1. *if* $r < 1$, *then* $\sum_{n=1}^{\infty} |a_n|$ *converges.*

2. if $r > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

3. $r = 1$, test fails.

Example 2.8. Find interval x such that $\sum_{n=1}^{\infty} \frac{n^2}{8^n} x^{3n}$ converges.

Solution. Let $a_n = \frac{n^2}{8^n} x^{3n}$. Then

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{n^2}{8^n}|x|^3} = \frac{|x|^3}{8}\sqrt[n]{n^2} \to \frac{|x|^3}{8} < 1, \qquad \text{when } |x| < 2.$$

Therefore, the series converges for all $x \in (-2, 2)$.

• We have discussed how to test if a series with non-negative terms converges. In general, testing the convergence of $\sum_{n=1}^{\infty} a_n$ with general a_n is very difficult. The following identity is a very useful tool.

Theorem 2.12. *Abel's Identity*

$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} s_k(b_k - b_{k+1}) + s_q b_q - s_{p-1} b_p$

where $s_k = \sum_{l=1}^{k} a_l$, $s_0 = 1$.

Proof. Notice that

$$\sum_{k=p}^{q} a_k b_k = \sum_{k=p}^{q} (s_k - s_{k-1}) b_k = \sum_{k=p}^{q-1} (s_k b_k - s_k b_{b+1}) + s_q b_q - s_{p-1} b_p = \sum_{k=p}^{q} s_k(b_k - b_{k+1}) + s_q b_q - s_{p-1} b_p.$$

The proof is complete. □

Theorem 2.13. *Test for alternating series*

Let $\{b_k\}_{k=1}^{\infty}$ is a decreasing sequence and $\lim_{n\to\infty} b_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

Proof. Apply Abel's identity with $a_n = (-1)^n$, $s_n = 0$ if n is even, -1 if n is odd.

Then the above theorem follows directly from the following more general theorem □

Theorem 2.14. *Let $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be two sequences of numbers such that*

(i) $\{s_n = \sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ is bounded;

(ii) $b_n \geq b_{n+1}$ for $n \geq 1$ and $b_n \to 0$ as $n \to \infty$.

Then $\sum_{k=1}^{\infty} a_k b_k$ *converges.*

Proof. For any $\epsilon > 0$, need to find N such that if $m, n > N$ then $|\sum_{k=n}^{m} a_k b_k| < \epsilon$. Since $\{s_n\}_{n=1}^{\infty}$ is bounded, there is $M > 0$ such that $|s_n| \le M$ for all $n \ge 1$, we have

$$\begin{aligned}
\left|\sum_{k=n}^{m} a_k b_k\right| &= \left|\sum_{k=n}^{m} s_k(b_k - b_{k+1}) + s_m b_m - s_{n-1} b_n\right| \\
&\le \sum_{k=n}^{m} |s_k|(b_k - b_{k+1}) + (|s_m| + |s_n|)(|b_m| + |b_n|) \\
&\le M(b_n - b_{m+1}) + 2M(|b_n| + |b_m|) \\
&\le 3M(|b_n| + |b_m|).
\end{aligned}$$

Since $\lim_{n\to\infty} b_n = 0$, there exists N such that if $n \ge N$, we have $|b_n| \le \epsilon/6M$. Therefore, when $m > n \ge N$, we have $|\sum_{k=n}^{m} a_k b_k| \le 6M \cdot \epsilon/6M = \epsilon$. Therefore, $\sum_{k=1}^{\infty} a_k b_k$ converges by the Cauchy criteria of convergence. □

Example 2.9. Determine if $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\ln(n+1)}$ converges.

Solution. Since $\ln(n+2) \ge \ln(n+1)$, $n = 1, 2, ...$, then $\dfrac{1}{\ln(n+2)} \le \dfrac{1}{\ln(n+1)}$, $n = 1, 2, ...$ and $\lim_{n\to\infty} \dfrac{1}{\ln(n+1)} = 0$. By the alternating series test, $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\ln(n+1)}$ converges.

2.4 Exercise

1. Prove the following series converge.
   (a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
   (b) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[3]{n}}$
2. Prove $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}^2$.
3. Investigate the behavior (convergence and divergence) of $\sum_{n=1}^{\infty} a_n$ if
   (a) $a_n = \sqrt{n+1} - \sqrt{n}$; (b) $a_n = (\sqrt{n+1} - \sqrt{n})/n$; (c) $a_n = (n^{1/n} - 1)^n$ and
   (d) $a_n = \frac{1}{1+z^n}$, where z is complex number.
4. If $a_n \ge 0$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.
5. If $\sum_{n=1}^{\infty} a_n$ converges and in $\{b_n\}$ is monotonic and bounded, prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

6. Suppose that $a_n > 0$ and $\sum_{n=1}^{\infty} a_n = \infty$.

(a) Prove that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges;

(b) What can be said about

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}, \quad \sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n},$$

(c) Let $s_n = \sum_{k=1}^{n} a_k$. Prove

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{s_n^2}$ converges.