Summer Jump-Start Program for Analysis, 2012

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1 Lecture 3: Power series and metric space, 8/15/2012

1.1 Power Series

For the power series $\sum_{n=0}^{\infty} a_n x^n$, the radius of convergence is

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Theorem 1.1. *The following statements hold.*

(i) $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely for* $x \in (-R, R)$*;*

(ii) $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely and uniformly for* $x \in [-r, r]$ *for any* $r < R$*;*

(iii) Let $f(x) =: \sum_{n=0}^{\infty} a_n x^n$*. Then* $f(x)$ *is both differentiable and integrable on* $(-R, R)$ *and*

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad x \in (-R, R)$$

and

$$\int_0^x f(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}, \quad x \in (-R, R).$$

(iv) For $x = R$ *and* $x = -R$*, it must be checked on a case-by-case basis.*

Example 1.1. Determine where the following power series converges $\sum_{n=1}^{\infty} 2^{-n} x^{n^2}$.

Solution (1) Find the radius of convergence:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n^2]{2^{-n}}} = 2.$$

(2) Examine the end points:

$x = \pm 1$. Since $\sum_{n=1}^{n} 2^{-n}$ and $\sum_{n=1}^{\infty} 2^{-n}(-1)^{n^2} = \sum_{n=1}^{\infty} 2^{n}(-1)^{n}$ both are convergent.

Example 1.2. Determine where $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}2^n} x^n$ converges.

Solution Since $R = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{\frac{1}{\sqrt{n}2^n}}} = 2$, by examining $x = -2, 2$, we have $\sum_{n=1}^{\infty} 2^n x^{n^2}$ converges on $(-2, 2]$.

1.2 Examples of power series of some elementary functions

1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $x \in (-1, 1)$;

   2) $\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$, $x \in (-1, 1)$;

   3) $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$, $x \in (-1, 1)$;

   4) $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $x \in (-\infty, \infty)$;

   5) $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, $x \in (-\infty, \infty)$;

   6) $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $x \in (-\infty, \infty)$;

1.3 Metric spaces

1.3.1 Definition of metrics

On $\mathbb{R}$, The distance between two numbers x and y is measured as a number:

$$d_e(x, y) = |x - y|, \quad x, y \in \mathbb{R}.$$

In $\mathbb{R}^n$, The distance between two points $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is measured as a number:

$$d_e(x, y) = |x - y| = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}, \quad x, y \in \mathbb{R}.$$

The $d_e(x, y)$ is so-called Euclidean metric which satisfies:

(a) $d_e(x, y) \geq 0$, $d_\epsilon(x, y) = 0$ if and only if $x = y$.

(b) $d_e(x, y) = d_\epsilon(y, x)$ (Symmetric).

(c) $d_e(x, z) \leq d_\epsilon(x, y) + d_\epsilon(y, z)$, $x, y, z \in \mathbb{R}$ (Triangle Inequality).

- **Motivation:**

  In $\mathbb{R}^n$, $d_e(x, y)$ may not be the unique way to measure distance between two points as our needs.

Definition 1.1. Let X be a set. A function $d : X \times X \to [0, \infty)$ is said to be a metric on X if d satisfies properties (a),(b) and (c) above. (X, d) is a called a metric space with distance function d.

1.3.2 Examples for metric spaces

On $\mathbb{R}^n = \mathbb{R} \times .... \times \mathbb{R} = \{(x_1, ..., x_n), x_j \in \mathbb{R}, 1 \leq j \leq n\}$, we have Euclidean metric d_e. In fact, one may assign other measurement functions on $\mathbb{R}^n$ such as:

Example 1.3. Let

$$d_p(x, y) = \Big(\sum_{j=1}^{n} |y_j - x_j|^p\Big)^{1/p}, \quad 1 \leq p < \infty.$$

Then d_p is metric on $\mathbb{R}^n$.

Example 1.4. Let A be $n \times n$ symmetric, positive definite matrix over $\mathbb{R}$. Then

$$d_A(x, y) = \sqrt{\langle A(y - x), y - x\rangle} = \sqrt{\sum_{i,j=1}^{n} a_{ij}(y_i - x_i)(y_j - x_j)}$$

Then d_A is a metric on $\mathbb{R}^n$.

Example 1.5. If (X, d) is a metric space and if we define $\tilde{d}(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$, $x, y \in X$. Prove that $(X, \tilde{d})$ is also a metric space.

Proof. (i) $\tilde{d}(x, y) = \dfrac{d(x, y)}{1 + d(x, y)} \geq 0$; $\tilde{d}(x, y) = 0 \iff d(x, y) = 0 \iff x = y$.
(ii) $\tilde{d}(x, y) = \dfrac{d(x, y)}{1 + d(x, y)} = \dfrac{d(y, x)}{1 + d(y, x)} = \tilde{d}(x, y)$.
(iii) For any $x, y, z \in X$, want to prove: $\tilde{d}(x, z) \leq \tilde{d}(x, y) + \tilde{d}(y, z)$.

We know that $d(x, z) \leq d(x, y) + d(y, z)$. Then:

$$\begin{aligned}
\tilde{d}(x, z) &= \frac{d(x, z)}{1 + d(x, z)} \\
&\leq \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \text{ since } \frac{x}{1 + x} \text{ is increasing in } x \geq 0 \\
&= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\
&\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\
&= \tilde{d}(x, y) + \tilde{d}(y, z)
\end{aligned}$$

Therefore, $(X, \tilde{d})$ is a metric space. □

Example 1.6. X is a set, and if we define the discrete metric as follows:

$d_0(x, y) = 0$ if $x = y$, $= 1$ if $x \neq y$. Prove that (X, d_0) is a metric space.

Proof. (i) $d_0(x, y) \geq 0$, and $d_0(x, y) = 0 \iff x = y$.

(ii) $d_0(x, y) = d_0(y, x) = 1$ or 0.

(iii) $d(x, z) = 0$ if $x = z$ or 1 if $x \neq z$.

For any $y \in X$, either $x \neq y$ or $z \neq y$ if $x \neq z$:

$d(x, y) + d(y, z) \geq 1 \geq d(x, z)$.

Therefore, (X, d_0) is a discrete metric. □

Definition 1.2. A ball $B(x, r)$ is a metric space (X, d) centered at x with radius $r > 0$ in the set $B(x, r) = \{y \in X, d(y, x) < r\}$.

Definition 1.3. Let $\{x_n\}_{n=1}^{\infty} \subset (X, d)$, $x \in X$. Then

- We say that the limit of a sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is x if $\lim_{n\to\infty} d(x_n, x) = 0$.
- We say that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence if for any $\epsilon > 0$, there exists an N such that if $m, n \geq N$ then $d(x_n, x_m) < \epsilon$.

Definition 1.4. A metric (X, d) is complete if every Cauchy sequence in X has a limit in X.

Example 1.7. (i) $(\mathbb{R}^n, d_\epsilon)$ is a complete metric space.

(ii) $(\mathbb{Q}, d_\epsilon)$ is not a complete metric space.

Definition 1.5. A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said contractive if there exists constant $c \in (0, 1)$ such that $d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n)$, $n = 2, 3, ...$

Example 1.8. Prove: Every contractive sequence in a metric space (X, d) must be a Cauchy sequence.

Proof. Since $\{x_n\}$ is contractive, there exists a constant $c \in (0, 1)$ such that

$$d(x_n, x_{n+1}) \leq cd(x_{n-1}, x_n), \quad n = 2, 3, ...$$

For any $m > n \geq 1$, by the triangle inequality, one has

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq \sum_{j-1}^{m-n} d(x_{n+j}, x_{n+j-1}) \\ &\leq \sum_{j=1}^{m-n} c^j d(x_n, x_{n-1}) \\ &\leq \sum_{j=1}^{m-n} c^j c^{n-2} d(x_2, x_1) \\ &= \frac{1 - c^{m-n}}{1-c} c^{n-1} d(x_2, x_1) \\ &\leq \frac{c^{n-1}}{(1-c)} d(x_2, x_1). \end{aligned}$$

For any $\epsilon > 0$ since $0 < c < 1$ and $\frac{c^{n-1}}{1-c} d(x_0, x_1) \to 0$ as $n \to \infty$. Then there is an N such that $c^n/(1-c) \cdot d(x_2, x_1) < \epsilon$. Therefore, when $m > n \geq N$, we have $d(x_m, x_n) < \epsilon$. This implies that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). □

Theorem 1.2. *In a complete metric space (X, d), every contractive sequence has a limit in X.*

Definition 1.6. Let (X, d) be a metric space. A map $F : X \to X$ is said to be a contractive map if there is $c \in (0, 1)$ such that

$$d(F(x), F(y)) \leq cd(x, y), \quad x, y \in X.$$

Theorem 1.3. *(Banach Fixed Point Theorem) In a complete metric space (X, d), every contractive map $F : X \to X$ has a fixed point in X.*

Proof. Let $x_0 \in X$ be any point. We define $x_n = F(x_{n-1})$ for $n = 1, 2, 3, \cdots$. Then $\{x_n\}_{n=1}^{\infty}$ is a contractive sequence, which has a limit $x \in X$. Since F must be continous, one can easily see that $x = F(x)$.

Example 1.9. If $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy in a metric (X, d), then prove $\{d(x_n, y_n)\}_{n=1}^{\infty}$ is a convergent sequence in $\mathbb{R}$.

Solution: For any m, n, compute: (I think I may have written out the indices incorrectly).

$$\begin{aligned} d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(x_m, y_n) - d(x_m, y_m) \\ &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) - d(x_m, y_m) \\ &= d(x_n, x_m) + d(y_m, y_n) \end{aligned}$$

By symmetry, one has

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n)$$

For any $\epsilon > 0$, since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy in (X, d), then $\exists N$ s.t. if $m, n \geq N$, then $d(x_m, x_n) < \epsilon/2$ and $d(y_m, y_n) < \epsilon/2$. Then, when $m, n \geq N$, $|d(x_n, y_n) - d(x_m, y_m)| < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore, $\{d(x_n, y_n)\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{d(x_n, y_n)\}_{n=1}^{\infty}$ is convergent in $\mathbb{R}$.

1.3.3 Union and intersection of sets

• Let (X, d) be a metric space and let $E \subset X$ be a set in X. Then we define:

• $\emptyset$ is the empty set.

• $E^c = \{x \in X : x \notin E\} = X \setminus E$ (complement set of E in X).

• Let A and B be two sets in X. Then

(1) $A \cup B = \{x : x \in A \text{ or } x \in B\}$ is the union of A and B.

(2) $A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$ is the intersection of A and B.

Theorem 1.4. *Let $\{A_\alpha\}_{\alpha \in I}$ be a family (finite or infinite) of sets in X. Then*

$$\left(\bigcup_{\alpha \in I} A_\alpha \right)^c = \bigcap_{\alpha \in I} (A_\alpha)^c$$

Proof. The proof follows from the below:

$$\begin{aligned} x \in (\textstyle\bigcup_{\alpha \in I} A_\alpha)^c \iff x \notin \textstyle\bigcup_{\alpha \in I} A_\alpha \iff x \notin A_\alpha \text{ for any } \alpha \in I \\ \iff x \in (A_\alpha)^c, \text{ for all } \alpha \in I \iff x \in \textstyle\bigcap_{\alpha \in I} (A_\alpha)^c. \end{aligned}$$

1.4 Exercise

1. Determine where the series $\sum_{n=1}^{\infty} \frac{1}{8^n \ln(n+1)} x^{3n}$ converges.

2. $X = C[0,1]$ be the set of all continuous functions on $[0,1]$. We define $d : X \times X \to [0,\infty)$ as follows:

$$d(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\}$$

Prove d is a metric on $C[0,1]$.

3. Let A and B be disjoint nonempty closed sets in a metric spaces X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}, \quad p \in X.$$

Show that f is a continuous function on X whose range lies in $[0,1]$, $f(p) = 0$ on A and $f(p) = 1$ on B.

4. Prove $d(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|$ define a metric in $\mathbb{R}^n$. Sketch the unit ball of $\mathbb{R}^2$ in this metric.

5. Let (X, d_0) be the discrete metric space. Find all ball $B(x,r)$ in (X, d_0).