Summer Jump-Start Program for Analysis, 2012

Song-Ying Li

1 Lecture 4: Set topology on metric spaces, 8/17/2012

Definition 1.1. Let (X, d) be a metric space; E is a subset of X. Then:

(i) $x \in E$ is an interior point of E if there is a $r > 0$ s.t. $B(x, r) \subset E$.

We let $\text{int}(E) = \{x \in E; x \text{ is an interior point of } E\}$.

(ii) E is an open set if every point of E is an interior point, i.e. $\text{int}(E) = E$.

(iii) $x \in E$ is an isolated point if there is $r > 0$ s.t. $B(x, r) \cap E = \{x\}$.

(iv) $x \in X$ is a limit point of E if for any $\epsilon > 0$ there is $y \in B(x, \epsilon) \cap E$, $y \neq x$.

Denote E' = all limit points of E.

(v) E is a closed set if $E' \subset E$;

$\overline{E} = E' \cup E =$ closure of E;

$E \backslash E'$ contains all isolated points of E.

(vi) E is a perfect set if $E = E'$.

(vii) Exterior of E is defined as: ext(E)=int(E^c)

(viii) Boundary of E is: $\partial E = E \setminus \text{int}(E) \setminus \text{ext}(E)$.

(ix) E is a compact set in (X, d) if for every open cover for E, there is a finite subcover.

In other word, If $\{U_\alpha : \alpha \in A\}$ is a family of open sets in X such that $E \subset \bigcup_{\alpha \in A} U_\alpha$, then there are finitely many U_{α_j} with $\alpha_1, \cdots, \alpha_n \in A$ such that $E \subset \cup_{j=1}^{n} U_{\alpha_j}$.

(x) $A, B \subset X$; A and B are separated if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

(xi) $E \subset X$ is a connected set if there are no non-empty separated sets A and B s.t. $E = A \cup B$.

Theorem 1.1. *(X, d) is a metric space:*

(i) $B(x, r)$ is an open set.

(ii) E is open if and only if E^c is closed.

(iii) The union of any open set is open.

(iv) Intersection of any closed sets is closed.

Example 1.1. Prove $B(x, r)$ is an open set.

Proof. To prove $B(x, r)$ is open, it suffices to prove: for any $y \in B(x, r)$ y is an interior point of $B(x, r)$. Let $\epsilon = r - d(x, y) > 0$, we claim $B(y, \epsilon) \subset B(x, r)$. For any $z \in B(y, \epsilon)$, WTS $d(z, x) < r$. Since $d(z, x) \leq d(z, y) + d(x, y) < \epsilon + d(x, y) = r - d(x, y) + d(x, y) = r$. Therefore, y is an interior point of $B(x, r)$ and thus $B(x, r)$ is open.

Example 1.2. Counterexamples for (iii) and (iv) in the previous theorem.

(i) $E_n = (-\infty, 1 - 1/n]$ $n = 1, 2, ...$ are closed, but $\cup_{n=1}^{\infty} E_n = (-\infty, 1)$ is not closed.

(ii) $E_n = (-\infty, 1 + 1/n)$ $n = 1, 2, ...$ are open, but $\cap E_n = (-\infty, 1]$ is not open.

Example 1.3. : Prove that in metric space (X, d), $E \subset X$, then the set of interior points in E, $\text{int}(E)$, is an open set.

Proof. : To prove $\text{int}(E)$ is open, it suffices to prove for any $x \in \text{int}(E)$, x is an interior point of $\text{int}(E)$.

For any $x_0 \in \text{int}(E)$, there is $r > 0$ such that $B(x_0, r) \subset E$. Then $B(x_0, r) = \text{int}(B(x_0, r)) \subset \text{int}(E)$. Therefore, x_0 is an interior point of $\text{int}(E)$. The proof is complete.

Example 1.4. Let $X = \mathbb{R}$ and $E = \mathbb{Q} \cap (0, 1)$. Find

(i) E' (all limit pts. of E),

(ii) $\text{int}(E)$ (all interior points of E).

Solution:

(i) We will show that $E' = [0, 1]$.

Let $x \in (0,1]$, for any $\epsilon > 0$, WTS there is a $r \in E \setminus \{x\}$ s.t. $|r - x| < \epsilon$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a $r \in ((x-\epsilon, x) \cap (x/2, x) \cap [0,1] \cap \mathbb{Q} \subset E$. By definition, $x \in E'$, Moreover, $0 = \lim_{n\to\infty} 1/n$, $1/n \in E$, so $0 \in E'$ and thus $E' = [0,1]$.

(ii) We will show that $\mathrm{int}(E) = \emptyset$.

For $x \in E$, WTS x is not an interior point in E, i.e. for $r > 0$, there is $y \notin E \cap (x - r, x + r)$. We choose $n \in \mathbb{N}$ with $n \geq 2(r+1)$, let $y = x + \dfrac{\sqrt{2}}{n}$. Then, $(y - r) = \sqrt{2}/n < r$ and $y \notin E$. Therefore, $x \notin \mathrm{int}(E)$. So, $\mathrm{int}(E) = \emptyset$.

1.1 Compact sets

1.1.1 Basic properties of compact sets

Recall that a set E of a metric space (X,d) is compact if each of its open covers of E has a finite subcover of E.

Theorem 1.2. *(i) Every compact set must be closed;*

(ii) Every closed subset of a compact set in a metric space is compact.

Proof. (i) E is a compact subset of (X,d). If E is not closed, then there is $x_0 \in E' \setminus E$. Choose a sequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that $d(x_n, x_0) \leq d(x_{n-1}, x_0)/2$. Then $\{x_n\}_{n=1}^{\infty}$ is closed set in X. Notice that

$$E \subset (X \setminus \{x_n : n \in \mathbb{N}\mathbb{N}\}) \cup_{n=1}^{\infty} B(x_n, \frac{d(x_n, x_0)}{4}).$$

Since $B(x_n, \frac{d(x_n,x_0)}{4}) \cap B(x_m, \frac{d(x_m,x_0)}{4}) = \emptyset$ if $m \neq n$. Therefore, there is no finite subcover for E. This is a contradiction with E is compact.

(ii) Let E is compact and F is closed subset of E. Then any open cover $\{O_\alpha : \alpha \in \Lambda\}$ for F, we have that $\{O_\alpha : \alpha \in \Lambda\} \cup \{X \setminus F\}$ is open cover of E. There is a finite subcover: $O_{\alpha_1}, \cdots, O_{\alpha_k}, X \setminus F$ for E. Therefore, F has a finite subcover.

Example 1.5. E_n is compact, $E_n \neq 0$, $E_{n+1} \subset E_n$, $n = 1, 2, ...$, show $\cap_{n=1}^{\infty} E_n \neq \emptyset$.

Proof. If $\cap_{n=1}^{\infty} E_n = \emptyset$, then $X = (\cap_{n=1}^{\infty} E_n)^c = \cup_{n=1}^{\infty} E_n^c$. E_n is compact, then E_n is closed and E_n^c is open. $E_1 \subset X \subset \cup_{nn=1}^{\infty} E_n^c$. Therefore, $\{E_n^c : n = 1, 2, ...\}$ is an open cover for E_1 and E_1 is compact, there is finite subcover $\{E_{n_j}^c : j = 1, ..., k\}$ with $n_1 \leq n_2 \leq \cdots \leq n_k$. Thus

$$E_{n_k} \subset E_1 \subset \cup_{j=1}^k E_{n_j}^c \subset E_{n_k}^c$$

This is a contradiction. Therefore $\cap_{n=1}^{\infty} E_n \neq \emptyset$. $\square$

1.1.2 Compact set in $\mathbb{R}^n$

When $X = \mathbb{R}^n$, we have better structure for a compact set.

Theorem 1.3. *Any cell $I_n = [a_1, b_1] \times .... \times [a_n, b_n]$ is a compact set of $\mathbb{R}^n$*

Proof. We prove the case $n = 1$. The proof is similar when $n > 1$. Let $\{O_\alpha, \alpha \in \Lambda\}$ be any open cover of $I = [a, b]$. Suppose that there is no finite sub cover for I, we try to get a contradiction. By the assumption, either $[a, \frac{a+b}{2}]$ or $[\frac{a+b}{2}, b]$ has no finite subcover. We denote the one which has no finite sub cover as $[a_1, b_1]$. Therefore,

$$a \leq a_1 \leq b_1 \leq b; \quad b_1 - a_1 = \frac{b-a}{2}$$

We divide interval $I^1 = [a_1, b_1]$ into two intervals $[a_1, \frac{a_1+b_1}{2}]$ and $[\frac{a_1+b_1}{2}, b_1]$. We conclude that one of them con not be covered by finite elements in $\{O_\alpha, \alpha \in \Lambda\}$. We denote the interval as $[a_2, b_2]$. Keeping the process, we will have $I^k = [a_k, b_k]$ which can not be covered by finite elements in $\{O_\alpha, \alpha \in \Lambda\}$ and satisfy the following property:

$$a_{k-1} \leq a_k \leq b_k \leq b_{k-1}; \quad b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2} = \frac{b-a}{2^k}, \quad a_0 = a, b_0 = b, \quad k = 1, \cdots$$

It is clear that $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are convergent sequences, they converge to the same point $x_0 \in I = [a, b]$. Therefore, there is a O_α with some $\alpha \in \Lambda$ such that $x_0 \in O_\alpha$. Since O_α is open, there is $\epsilon > 0$ such that $x_0 - \epsilon, x_0 + \epsilon) \subset O_\alpha$. When $2^{-k} < \epsilon$, we have that $I^k \subset (x_0 - \epsilon, x_0 + \epsilon) \subset O_\alpha$. This contradicts with I^k can not be covered by finite elements in $\{O_\alpha : \alpha \in \Lambda\}$. Therefore, $I = [a, b]$ is compact. $\square$

Corollary 1.4. *(Contractive net cell theorem) If* $...I_m \subset ... \subset I_1 \subset I$ *and* $dia(I_m) \to 0$ *as* $m \to \infty$*, then* $\cap_{m=1}^{\infty} I_m = \{x_0\}$.

Remark 1.1. If I_m is not compact, the theorem may fail.

Example 1.6.

(i) $E_n = (0, 1/n)$ and $E_{n+1} \subset E_n$, but $\cap_{n=1}^{\infty} E_n = \emptyset$.

(ii) $E_n = [n, \infty)$ closed and $E_{n+1} \subset E_n$, but $\cap_{n=1}^{\infty} E_n = \emptyset$.

Theorem 1.5. *Heine-Borel Theorem*

Let $E \subset \mathbb{R}^n$*. Then* E *is compact* $\iff$ E *is bounded and closed.*

Proof. A brief proof: Notice that

$$E = \cup\{B(x, 1) : x \in E\}$$

Then E is compact, it must be bounded and closed. Conversely, since E is bounded in $\mathbb{R}^n$, there is $M > 0$ such that

$$E \subset [-M, M]^n.$$

Notice that $[-M, M]^n$ is compact and E is closed, so E is compact. ◻

Theorem 1.6. *(Weierstrass theorem) Every bounded infinite subset* E *of* $\mathbb{R}^k$ *has a limit point in* $\mathbb{R}^k$.

Proof. Since E is bounded, there is $M > 0$ such that $E \subset I = [-M, M]^n$. Using the idea of the proof of Theorem 4.2, one can construct cells I^k such that

i) $I^k \subset I^{k-1}$, $I_0 = I$, $k = 1, 2, 3, \cdots$;

ii) $\text{dia}(I^k) \leq \sqrt{n} 2M$

iii) Each I^k contains infinite many points of E.

Then $\{x_0\} = \cap I^k$ which is a limit point of E. ◻

1.2 Connected and convex sets

Let (X, d) ne a metric space. Then E is connected if $E \neq A \cup B$ with $A \neq \emptyset$, $B \neq \emptyset$, $\bar{A} \cap B = \emptyset$, $A \cap \bar{B} = \emptyset$. An empty connected set in $\mathbb{R}$ is very simple:

Theorem 1.7. *A connected set E in $\mathbb{R}$ must be an interval (i.e. $x_1, x_2 \in E_1$, $x_1 < x_2$ such that $(x_1, x_2) \subset E \subset [x_1, x_2] \subset E$).*

Proof. Suppose the theorem is not true: there are two points $x, y \in E$, $x < y$ but there is $z \neq E$ and $x < z < y$. We try to get a contradiction.

Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E)$. Then $E = A \cup B$. Moreover, $\bar{A} \subset (-\infty, z] \cap \bar{E}$ and $\bar{B} \subset [z, \infty) \cap \bar{E}$. Thus, $\bar{A} \cap B = \emptyset$ and $A \cap B = \emptyset$.

Since $x \in A$, $y \in B$, one has A, B are not empty separated sets in $\mathbb{R}$ and $E = A \cup B$. Therefore, E is not connected. This is a contradiction. □

Definition 1.2. (i) For any $x, y \in \mathbb{R}^n$, we define a line segment $[x, y] = \{\lambda x + (1 - \lambda) y : \lambda = [0, 1]\}$.

(ii) $E \subset \mathbb{R}^n$ is convex if $[x, y] \subset E$ for all $x, y \in E$.

Example 1.7. (i) $B(0, r)$ and any cells in $\mathbb{R}^n$ are convex;

(ii) If $D_1, ..., D_m$ are convex in $\mathbb{R}^n$, then $D_1 \times D_2 \times ... \times D_n$ is convex in $\mathbb{R}^{n \times m}$.

1.3 Exercise

1. Let E be a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf\{d(x, y) : y \in E\}.$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$;

(b) Prove that ρ_E is uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y), \quad x, y \in X.$$

2. Let $x \in \mathbb{R}^n$. Prove the unit ball $B(x, r)$ centered at x with radius r is a convex set

3. Prove the polydisc $D(0, r_1) \times D(0, r_2) \times \cdots \times D(0, r_n)$ is a convex set in $\mathbb{R}^n$, where $D(0, r_j)$ is the disc in $\mathbb{R}^2$.

4. Prove the set of irrational number is uncountable.

5. Let L_n be a line in $\mathbb{R}^2$ for $n = 1, 2, 3, \cdot$. Prove $\cup_{n=1}^{\infty} L_n \neq \mathbb{R}^2$.

6. Construct a set of real numbers which has exactly three limit points.

7. Let $A_1, \cdots, A_n, \cdots$ be subsets of a metric space X. Prove

(a) If $B_n = \cup_{k=1}^{n} A_k$ then $\overline{B_n} = \cup_{k=1}^{n} \overline{A_n}$;

(b) If $B = \cup_{k=1}^{\infty} A_k$ then $\cup_{k=1} \infty \overline{A}_k \subset \overline{B}$. Give an example showing that $\cup_{k=1}^{\infty} \overline{A}_k$ is proper subset of $\overline{B}$.

8. Let $E = \mathbb{Q}$ the set of the rational numbers. Find

(a) the set of all interior point of E, (b) the set of all limit points of E; (c) boundary of E.

9. For $x, y \in \mathbb{R}$, define:

$$d_1(x, y) = (x - y)^2, \ d_2(x, y) = \sqrt{|x - y|}, \ d_3(x, y) = |x^2 - y^2|, \ d_4(x, y) = |x - 2y|$$

and $d_5(x, y) = \frac{|x-y|}{1+|x-y|}$. Determine, for each of these, whether it is a metric or not.

10. Let $K = \{1/n, n \in \mathbb{N}\} \cup \{0\}$. Prove K is compact subset of $\mathbb{R}$.