Summer Jump-Start Program for Analysis, 2012

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1 Lecture 5: Limits and Continuities of Functions

1.1 Definitions and examples for limit and continuity

Definition 1.1 *Let* (X, d_1) *and* (Y, d_2) *be two metric spaces.* $E \subset X$ *and* $f : X \to Y$ *be a function (map). Let* $x_0 \in E'$, $y_0 \in Y$. *Then*

(i) We say that $f(x)$ *has limit* y_0 *as* $x \to x_0$ *or* $\lim_{x\to x_0} f(x) = y_0$ *if for any* $\epsilon > 0$, *there is* $\delta > 0$ *such that if* $d_1(x, x_0) < \delta$ *and* $x \in E$ *then* $d_2(f(x), y_0) < \epsilon$;

(ii) We say that f *is continuous at* x_0 *if* $x_0 \in E$ *and* $\lim_{x\to x_0} f(x) = f(x_0)$;

(iii) We say that f *is discontinuous at* x_0 *if* f *is not continuous at* x_0.

We will study $X = \mathbb{R}^n$ and $Y = \mathbb{R}$ first.

• Basis properties for limits.

THEOREM 1.2 *If* $f, g : \mathbb{R}^n \to \mathbb{R}$ *are two functions.* $x_0 \in \mathbb{R}^n$. *Assume that* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, then*

(i) $\lim_{x\to x_0}(f(x) \pm g(x)) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$;

(ii) $\lim_{x\to x_0}(f(x)g(x)) = (\lim_{x\to x_0} f(x))(\lim_{x\to x_0} g(x))$;

(iii) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$ *if* $\lim_{x\to x_0} g(x) \neq 0$.

EXAMPLE 1 *Let*

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Discuss the limit and continuity of f *on* $\mathbb{R}^2$.

Solution. For any $(x_0, y_0) \in \mathbb{R}^2$. We divide it into two cases.

Case 1: $(x_0, y_0) \neq (0, 0)$ or $x_0^2 + y_0^2$.

If (x, y) is closed to (x_0, y_0), then $(x, y) \neq (0, 0)$. Thus,

$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = \lim_{x\to x_0, y\to y_0} \frac{xy}{x^2+y^2} = \frac{x_0y_0}{x_0^2+y_0^2} = f(x_0, y_0)$.

Case 2 $(x_0, y_0) = (0, 0)$.

$$\lim_{y=x\to 0} f(x, y) = \lim_{x\to 0} \frac{xx}{x^2 + x^2} = \frac{1}{2}$$

and

$$\lim_{x\to 0, y=0} f(x, y) = \lim_{x\to 0} \frac{0}{x^2} = 0.$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. Therefore, f is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

EXAMPLE 2 *Let*

$$f(x,y) = \begin{cases} \frac{x^3 - y^2 x}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Prove f is continuous on $\mathbb{R}^2$.

EXAMPLE 3 *Find the largest set C in $\mathbb{R}^2$ such that. f is continuous, where*

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2 + y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Solution.

We claim $C = \mathbb{R}^2 \setminus \{(0,0)\}$.

For any $(x_0, y_0) \in C$, since $x_0^2 + y_0^2 \neq 0$, we have

$$\lim_{(x \to 0, y=0} f(x,y) \quad = \lim_{(x\to 0} \frac{xy^2}{x^2+y^4} = \frac{x_0 y_0^2}{x_0^2 + y_0^4} = f(x_0, y_0)$$

So, f is continuous at (x_0, y_0).

Notice that $\lim_{x\to 0, y=0} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{0}{x^2} = 0$ and $\lim_{x=y^2\to 0} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^2 y^2}{y^4+y^4} = 1/2$. Therefore, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist. Thus, f is cont. is $c = \mathbb{R}^2 \setminus \{(0,0\}$.

EXAMPLE 4 *Let*

$$f(x,y) = \begin{cases} \frac{\sin(x^2 - y^2)}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Discuss the limit and continuity of f on $\mathbb{R}^2$.

EXAMPLE 5 *Construct a function $f : \mathbb{R} \to \mathbb{R}$ that f is discontinuous at every point of $\mathbb{R}$.*

Solution We claim that the Dirichlet function

$$D(x) = f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ irrational} \end{cases}$$

is discontinuous at every point of $\mathbb{R}$.

This is easily followed by the fact that $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ both are dense in $\mathbb{R}$.

EXAMPLE 6 *Construct a function f on $[0,1]$ such that f is continuous at every irrational point and discontinuous at every rational point in $[0,1]$.*

Solution The following Riemann function

$$R(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1/n & \text{if } x = m/n,\ m < n \\ 1 & \text{if } x = 0 \text{ or } 1 \end{cases}$$

is continuous at every irrational point and discontinuous at every rational point in $[0, 1]$.

Proof. *Case 1.* Let $x_0 \in (0, 1)$ be any irrational in $[0, 1]$.

We know $R(x_0) = 0$. For $\epsilon > 0$, we need to find $\delta > 0$ such that if $|x - x_0| < \delta$, then $|R(x) - R(x_0)| = R(x) < \epsilon$.

We consider the set $E_\epsilon =: \{\gamma = \frac{m}{n} \in (0, 1) : (m, n) = 1 \text{ and } \frac{1}{n} \geq \epsilon\}$. It is obvious E_ϵ is finite set, say $E_\epsilon = \{\gamma_j = \frac{p_j}{q_j}, j = 1, \cdots, k\}$. Let $\delta = \min\{|\gamma_j - x_0| : j = 1, \cdots, k\}$. Then if $|x - x_0| < \delta$ then $R(x)| < \epsilon$. Therefore, $R(x)$ is continuous at x_0.

Case 2. Let $x_0 = \frac{m}{n}$, $(m, n) = 1$, $0 < m < n$ be any rational point in $(0, 1)$. Then, $R(x_0) = \frac{1}{n}$. Choose $\epsilon_0 = 1/n$, f or $\delta > 0$, there is irrational $x_\delta \in (x_0 - \delta, x_0 + \delta)$ since $\mathbb{R} \setminus \mathbb{Q}$ is dense.

Then $|R(x_\delta) - R(x_0)| = 1/n = \epsilon_0$. By definition, $R(x)$ is not continuous at x_0. Similarly, one can prove $R(x)$ is discontinuous at $x_0 = 0$ or 1 since $R(0) = R(1) = 1$. □

Definition 1.3 *We say that a function $f(x)$ on (a, b) is increasing (non-decreasing) if $f(x_1) \leq f(x_2)$ when $x_1 < x_2$.*

THEOREM 1.4 *Let $f(x)$ be an increasing function on (a, b). Let $D_f = \{x_0 \in (a, b) : f$ is discont. at $x_0\}$. Then, D_f is at most countable.*

Proof. Let $x_0 \in D_f$. Since f is incasing, one has:

$\lim_{x \to x_0^+}(x) = f(x_0^+)$ exists and $\lim_{x \to x_0^-}(x) = f(x_0^-)$ exists.

Since f is increasing and discontinuous at x_0, one has $f(x_0^-) < f(x_0 + 1)$. Let

$$I_{x_0} = (f(x_0^-), f(x_0^+)).$$

Then, $f(x_0+) \leq f(x_1-)$ if $x_0 < x_1$ and $I_{x_0} \cap I_{x_1} = \emptyset$ for $x_0, x_1 \in D_f$ with $x_0 \neq x_1$.

Choose a rational number $\gamma_x \in I_x$ for any $x \in D_f$. Then $\gamma_{x_0} < \gamma_{x_1}$ if $x_0, x_1 \in D_f$ and $x_0 < x_1$. Then $\{\gamma_{x_0} : x_0 \in D_f\}$ is a subset of $\mathbb{Q}$, it is at most countable. D_f is at most countable set. □

EXAMPLE 7 *An abstract way to define e^x:*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x + y) = f(x)f(y)$, $x, y \in \mathbb{R}$ and $f(1) = e$. Then $f(x) = e^x$ for $x \in \mathbb{R}$.

Proof. Since $f(1) = f(0)f(1)$, one has $f(0) = 1$. Since $f(0) = f(-1)f(1)$, one has $f(-1) = e^{-1}$. Thus

$$f(m) = f(1)f(m-1) = f(1)^m = e^m, m \in \mathbb{N}$$

Notice that $f(1) = f(n/n) = (f(1/n))^n$, so $f(1/n) = e^{1/n}$ and $f(m/n) = f(1/n)^m = e^{m/n}$ when $m, n \in \mathbb{N}$. Notice that $1 = f(0) = f(x + (-x)) = f(x) + f(-x)$, we have $f(-x) = 1/f(x)$. Therefore, $f(r) = e^r$ for all $r \in \mathbb{Q}$. Since f is continuous on $\mathbb{R}$ and $\mathbb{Q}$ is dense in $\mathbb{R}$. If $x = \lim_{n\to\infty} \gamma_n$ with $\gamma_n \in \mathbb{Q}$ then

$$f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} e^{\gamma_n} = e^x.$$

□

EXAMPLE 8 *An abstract way to define* $\ln x$*:*

Let $f : (0, \infty) \to \mathbb{R}$ *be a continuous function satisfying* $f(xy) = f(x)+f(y)$, $x, y \in (0, \infty)$ *and* $f(e) = 1$. *Then* $f(x) = \ln x$ *for* $x \in \mathbb{R}$.

Definition 1.5 *Let* f *be a function on a convex subset* E *in* $\mathbb{R}^n$. *We say that* f *is convex if*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y), \qquad \text{for any } x, y \in E \text{ and } \lambda \in (0,1).$$

EXAMPLE 9 f *is convex on a convex set in* $\mathbb{R}^n$ *if and only if*

$$f(\frac{x+y}{2}) \le \frac{f(x)+f(y)}{2}, \quad x, y \in E.$$

THEOREM 1.6 *If* f *is a convex function on a convex subset* E *of* $\mathbb{R}^n$, *then* f *is continuous on* E.

Proof. We consider $n = 1$ case. Let $E = [a, b]$ and $x_0 \in (a, b)$. We show $\lim_{x\to x_0} f(x) = f(x_0)$. Notice that

$$f(x) = f(\lambda x_0 + (1-\lambda)b) \le \lambda f(x_0) + (1-\lambda)f(b)$$

Then

$$f(x) - f(x_0) \le (1-\lambda)(f(b) - f(x_0)),$$

and $x = \lambda x_0 + (1-\lambda)b \to x_0^+$ if and only if $\lambda \to 1^-$. Therefore, $\limsup_{x\to x_0^+}(f(x) - f(x_0)) \le 0$. If $x \in (x_0, b)$, we have

$$x_0 = \lambda x + (-\lambda)a$$

and

$$f(x_0) \le \lambda f(x) + (1-\lambda)f(a)$$

Thus

$$f(x_0) - f(x) \le (1-\lambda)(f(a) - f(x))$$

When $x \to x_0^+$, one has $\lambda \to 1^-$. Thus

$$\limsup_{x \to x_0^+} (f(x_0) - f(x)) \le 0.$$

Therefore, $f(x_0^+) = f(x_0)$. Similarly $f(x_0^-) = f(x_0)$. So f is continuous at x_0, and so on E.

1.2 Properties of continuous functions

We first list below important properties of a continuous function:
(1). Continuous function preserves the sign:

Let (X, d) be a metric space. If $f(x)$ is continuous at $x_0 \in X$ and $f(x_0) > 0$, then there is $\delta > 0$ such that $f(x) \ge \frac{f(x_0)}{2} > 0$ for all $x \in B(x_0, \delta)$.

Proof. By definition, choosing $\epsilon = \frac{f(x_0)}{2} > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon = \frac{f(x_0)}{2}, \quad x \in B(x_0, \delta).$$

Thus, $f(x) - f(x_0) > -\frac{f(x_0)}{2}$, and so $f(x) > \frac{f(x_0)}{2} > 0$ for all $x \in B(x_0, \delta)$.
(2). An equivalent definition for continuous function:

Let (X, d_1) and (Y, d_2) be two metric spaces. Then $f : X \to Y$ is continuous if and only for any open set $V \subset Y$, we have $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is open in X.

Proof. "$\Rightarrow$" Assume f is continuous from $X \to Y$.

For any open set $V \subset Y$, we want to prove $f^{-1}(V)$ is open in X. Let $x_0 \in f^{-1}(V)$. Then $y_0 = f(x_0) \in V$ and V is open, so there is $\epsilon > 0$ such that $B_Y(y_0, \epsilon) \subset V$. Since f is continuous at x_0, there is $\delta > 0$ such that $f(B_X(x_0, \delta)) \subset B_Y(y_0, \epsilon) \subset V$. Therefore, $B(x_0, \delta) \subset f^{-1}(V)$. So, $f^{-1}(V)$ is open.

"$\Leftarrow$" Assume $f^{-1}(V)$ is open in X when V is open in Y. Show f is cont. from $X \to Y$.

For any $x_0 \in X$, $f(x_0) \in Y$ and for any $\epsilon > 0$, $B_Y(f(x_0), \epsilon)$ is open set in Y. Then $f^{-1}(B_Y(B(f(x_0), \epsilon))$ is open in X containing x_0. So, there is $\delta > 0$ so that $B_X(x_0, \delta) \subset f^{-1}(B_Y(B(f(x_0), \epsilon))$ and $f(B_X(x_0, \delta)) \subset B_Y(f(x_0), \epsilon)$. Thus, f is continuous at x_0.
(3). Existence of maximum and minimum.

THEOREM 1.7 *Let (X, d) be a complete metric space and let K be a compact set in X. Let $F : K \to \mathbb{R}$ be continuous function. Then there are $x_0, y_0 \in K$ s.t. $f(x_0) \le f(x) \le f(y_0)$, $x \in K$.*

Proof. Let

$$M = \sup\{f(x) : x \in K\} \quad \text{and } m = \inf\{f(x) : x \in K\}.$$

By definition, there are sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in K such that

$$M = \lim_{n\to\infty} f(y_n) \quad \text{and} \lim_{n\to\infty} f(x_n) = m.$$

Since K is compact and X is complete, there are subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ and $x_0, y_0 \in K$ such that

$$\lim_{k\to\infty} x_{n_k} = x_0 \quad \text{and} \lim_{k\to\infty} y_{n_k} = y_0.$$

Since f is continuous at x_0 and y_0, we have

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = m \quad \text{and } f(y_0) = \lim_{k\to\infty} f(y_{n_k}) = M.$$

These prove the theorem. $\square$

(4). Intermediate Value Theorem

THEOREM 1.8 *Let $f; [a,b] \to \mathbb{R}$ be continuous. Then $f([a,b]) = [m, M]$, where*

$$M = \max\{f(x) : x \in K\} \quad \textit{and } m = min\{f(x) : x \in K\}.$$

In other words, for any $x_1 < x_2$ in $[a,b]$ and c is between $f(x_1)$ and $f(x_2)$, there is a $x_0 \in (x_1, x_2)$ such that $f(x_0) = c$.

EXAMPLE 10 *Let $f(x) = -2x^3 + 100x^2 - x + 100$. Then f has a real root.*

Solution: Since

$$\lim_{x\to+\infty} f(x) = -\infty \text{ and } \lim_{x\to-\infty} f(x) = +\infty.$$

There is $M >> 1$ such that $f(M) < 0$ and $f(-M) > 0$. By the Intermediate-Value-Theorem, there is $x_0 \in (-M, M)$ such that $f(x_0) = 0$.

1.3 Exercises

1. Suppose f is a real-valued function on $\mathbb{R}$ which satisfies: $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$ for every $x \in \mathbb{R}$. Does this implies that f is continuous.

2. Write $\mathbb{R}^2 = A \cup B$ so that f is continuous on A, discontinuous on B, where f is given as follows:
(i) $f(x,y) = (x^3 - y^2)/(x^2 - y^2)$ if $x^2 \neq y^2$, otherwise, $f(x,y) = 0$;
(ii) $f(x,y) = xy^2/(x^2 + y^4)$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$;
(iii) $f(x,y) = xy^2/(x^2 + y^6)$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

3. Let (X, d_X) and (Y, d_Y) be two metric spaces. If $f : X \to Y$ is continuous, prove $f(\overline{E}) \subset \overline{f(E)}$ for every subset E of X.

4. Let (X, d_X) and $Y, d_Y)$ be two metric spaces. Let $f, g : X \to Y$ be continuous. Prove

a) For each $q \in Y$, let $E_q = \{x \in X : f(x) = q\}$, prove E_q is closed in X;

b) Let $E = \{x \in X : f(x) = g(x)\}$. Then E is dense in X if and only if $f(x) \equiv g(x)$ on X.

5. Let $f(x) = 0$ when x is irrational, $f(0) = 0$ and let $f(x) = 1/n$ if $x = m/n$ with $(|m|, n) = 1$ where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.

a) Prove $f(x)$ is continuous at every irrational point or 0;

b) Prove $f(x)$ is discontinuous at $r = m/n \neq 0$.

6. Let (X, d) be a metric space, and let $f : X \to \mathbb{R}$ be a function. The graph of f on E is the set $G_E(f) = \{(x, f(x)) : x \in E\}$. Suppose E is compact. Prove that f is continuous on E if and only if $G_E(f)$ is compact in $X \times \mathbb{R}$.

7. Let A and B be disjoint nonempty closed sets in a metric spaces X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}, \quad p \in X.$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, $f(p) = 0$ on A and $f(p) = 1$ on B.

8. Let $f : [0, 1] \to [0, 1]$ be a continuous map. Prove that there is $x \in [0, 1]$ so that $f(x) = x$.

9. Let $[x]$ denote the largest integer contained in x, that is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?

10. Let $f(x)$ be a function defined on (a, b). Let $x_0 \in (a, b)$, we write $f(x_0-) = \lim_{x \to x_0, x < x_0} f(x)$ if the limit exists, and $f(x_0+) = \lim_{x \to x_0, x > x_0} f(x)$ if the limit exists. We say that x_0 has a simple discontinuity at x_0 if $f(x_0-) < f(x_0+)$. Prove the set of simple discontinuity of f in (a, b) is at most countable.

11. Let $f : X \to Y$ be a map, we say that f is an open mapping if $f(V)$ is open in Y when V is open in X. Prove every continuous open mapping from $\mathbb{R}$ to $\mathbb{R}$ is monotone.

12. Let $F : X \to Y$ be a function. Prove $f^{-1}(F^c) = (f^{-1}(F))^c$ for any $F \subset Y$. Here $F^c = Y \setminus F$ and $E^c = X \setminus E$ if $E \subset X$.