Summer Jump-Start Program for Analysis, 2012

Song-Ying Li

1 Lecture 6: Uniformly continuity and sequence of functions

1.1 Uniform Continuity

Definition 1.1 *Let (X, d_1) and (Y, d_2) are metric spaces and $K \subset X$. $f : K \to Y$ is said to be uniformly continuous on K: if for any $\epsilon > 0$, there is $\delta > 0$ such that $d_2(f(x), f(y)) < \epsilon$ whenever $d_1(x, y) < \delta$ for all $x, y \in K$.*

Note: We can think of uniformly continuous as a group (global property) action whereas cont. is single (local property) action.

THEOREM 1.2 *If f is uniformly continuous on K, then $f : K \to Y$ is continuous on K, but converse is not true.*

A counterexample: $f(x) = x^2$ is continuous on $(-\infty, \infty)$, but it is not uniformly continuous on $(-\infty, \infty)$.

Idea: we need to prove that there is $\epsilon_0 > 0$ s.t. for any $\delta > 0$, there are $x, y \in (-\infty, \infty)$ with $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon_0$.

Proof. Notice that: $|f(x)-f(y)| = |x^2-y^2| = |(x+y)(x-y)| = |x+y||x-y| = n \cdot \delta \geq \epsilon_0$. Let $\epsilon_0 = 1$. For any $\delta > 0$, let $x_\delta = 1/\delta$, $y_\delta = 1/\delta + \delta/2$. Then, $|x_\delta - y_\delta| = \delta/2 < \delta$. But, $|f(x_\delta) - f(y_\delta)| = |(x_\delta + y_\delta)||x_\delta - y_\delta| = (1/\delta + 1/\delta + \delta)\delta/2 \geq 2/\delta \cdot \delta/2 = 1 = \epsilon_0$. Therefore, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. ∎

THEOREM 1.3 *Let (X, d) be a complete metric space and $K \subset X$ is compact set. If $f : K \to Y$ is continuous on K, then f is uniformly continuous on K.*

Proof. If f is not uniformly continuous on K, then there is $\epsilon_0 > 0$ such that for any $\delta > 0$, there are x_δ, $y_\delta \in K$ such that

$$d_X(x_\delta, y_\delta) < \delta \quad \text{and} \quad d_y(f(x_\delta), f(y_\delta)) \geq \epsilon_0.$$

For $\delta = 1/n$, there are $x_n, y_n \in K$ such that

$$d_X(x_n, y_n) < 1/n \quad \text{and} \quad d_Y(f(x_n), f(y_n)) \geq \epsilon_0, \quad n = 1, 2, \cdots$$

Since K is compact and X is complete and $d_X(x_n, y_n) \to 0$ as $n \to \infty$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a subsequence $\{y_{n_k}\}$ and $x \in X$ such that $\lim_{k\to\infty} x_{n_k} = x = \lim_{k\to\infty} y_{n_k}$. Since f is continuous at x, we have

$$0 < \epsilon_0 \leq d_Y(f(x_{n_k}), f(y_{n_k})) \leq d_Y(f(x_{n_k}, f(x)) + d_Y(f(x), f(y_{n_k})) \to 0 \quad \text{as } k \to \infty.$$

This is a contradiction. So f is uniformly continuous on K. ∎

• **Question**: How to test a function is uniformly continuous ?
1.) f is continumus on a compact set K, then f is uniformly continuous on K.

2.) Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$. If $||\nabla f(x)|| = \sqrt{\sum_{j=1}^n \left|\frac{\partial f}{\partial x_j}(x)\right|^2}$ is bounded on D, then f must be uniformly continuous on D.

When $n = 1$, $|f(x) - f(y)| = |f'(\xi)||x - y| \le M|x - y|$, $x, y \in D = (a, b)$.

For any $\epsilon > 0$, let $\delta = \epsilon/M$ if $|x - y| > \delta$, then $|f(x) - f(y)| \le M|x - y| \le M \cdot \epsilon/M = \epsilon$. So f is uniformly continuous.

1.2 Examples for uniformly continuous functions

EXAMPLE 1 *Show $f(x) = \sqrt{x}$ is uniformly continuous on $(0, \infty)$.*

Proof. For any $\epsilon > 0$.

(i) Since $f(x) = \sqrt{x}$ is contiunus on $[0, 2]$, and $[0, 2]$ is compact, $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 2]$, so there is $\delta_1 > 0$ such that if $x_1, x_2 \in [0, 2]$ and $|x_1 - x_2| < \delta_1$, we have $|f(x_1) - f(x_2)| < \epsilon$.

(ii) Let $\delta_2 = \frac{\epsilon}{2}$. Since $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}}$ and $|f'(x)| \le \frac{1}{2}$, $x \in [1, \infty)$. By the Mean Value Theorem: $|f(x_1) - f(x_2)| = f'(c)|x_1 - x_2| \le \frac{1}{2}|x_1 - x_2| < \epsilon$ whenever $x_1, x_2 \in [1, \infty)$ and $|x_1 - x_2| < \delta_2$.

(iii) Choose $\delta = \min\{\delta_1, \delta_2, 1\}$, if $x_1, x_2 \in (0, \infty)$ and $|x_1 - x_2| < \delta$ then either $x_1, x_2 \in (0, 2]$ and $|x_1 - x_2| < \delta_1$ or $x_1, x_2 \in [1, \infty)$ and $|x_1 - x_2| < \delta_2$. Therefore, in any cases, we have if $|x_1, x_2| < \delta$ then $|f(x_1) - f(x_2)| < \epsilon$ for all $x_1, x_2 \in (0, \infty)$. This proves f is uniformly continuous on $(0, \infty)$. ∎

Remark: An easier way to $\sqrt{x}$ is uniformly continuous on $(0, \infty)$ is as follows: For $x_1, x_0 \in (0, \infty)$ and $x_1 < x_2$, one has

$$|\sqrt{x_2} - \sqrt{x_1}| = \frac{x_2 - x_1}{\sqrt{x_1} + \sqrt{x_2}} \le \sqrt{x_2 - x_1}$$

For any $\epsilon > 0$, choose $\delta = \epsilon^2$. When $x_1, x_2 \in (0, \infty)$ and $|x_1 - x_2| < \delta$, we have $|\sqrt{x_2} - \sqrt{x_1}| \le \sqrt{|x_2 - x_1|} < \epsilon$.. ∎

Similar argument shows:

EXAMPLE 2 *$f(x)$ is uniformly on $[\delta_1, \infty)$ ($\delta_j > 0$ is fixed), and $f(x)$ is uniformly on $[0, \delta_2]$ and $\delta_2 > \delta_1$. Then $f(x)$ is uniformly continuous on $[0, \infty)$.*

EXAMPLE 3 *Show $f(x) = log(1 + ||x||^2)$ is uniformly on $\mathbb{R}^n$.*

Proof. Since $\frac{\partial f}{\partial x_j}(x) = \frac{2x_j}{1+||x||^2}$, for any $x \in \mathbb{R}^n$, one has

$$||\nabla f(x)|| = \left(\sum_{j=1}^n \left|\frac{\partial f}{\partial x_j}\right|^2\right)^{1/2} = \sqrt{\sum_{n=1}^n \frac{4x_j^2}{(1 + ||x||^2)^2}} = \sqrt{\frac{4||x||^2}{(1 + ||x||^2)^2}} = \frac{2||x||}{(1 + ||x||)^2} \le 1.$$

Then we have f is uniformly continuous on $\mathbb{R}^n$. In fact:

$$\begin{aligned} |f(x) - f(y)| &= \left| \left(\frac{\partial}{\partial t} f(tx + (1-t)y) \right) |_\theta \cdot (1-0) \right| \\ &= |\nabla f(\theta x + (1-\theta)y) \cdot (x-y)| \\ &\leq ||\nabla f(\theta x + (1-\theta)y)|| ||x-y|| \\ &\leq ||x-y|| \end{aligned}$$

Thus, f is uniformly continuous on $\mathbb{R}^n$. ∎

THEOREM 1.4 *Let (X, d) be a metric space and let $K \subset X$. If $f(x)$ is a uniformly continuous function on K, then $f(x)$ can be extended as a uniformly continuous function on $\overline{K}$.*

Proof. For $x_0 \in K' \setminus K$. How to define $f(x_0)$? Choose a $\{x_n\}_{n=1}^\infty \subset K$ and $x_n \to x_0$ as $n \to \infty$. Since $f : K \to \mathbb{R}$ is uniformly continuous, one can easily see that $\{f(x_n)\}$ is Cauchy sequence in $\mathbb{R}$. So, there is a number, say $f(x_0)$ such that $\lim_{n\to\infty} f(x_n) = f(x_0)$. In order to prove $f(x_0)$ is well-defined, we need to prove for any sequence $\{y_n\} \subset K$ and $y_n \to x_0$ as $n \to \infty$, one has $\lim_{n\to\infty} f(y_n) = f(x_0)$. This can be followed from below:

$$|f(y_n) - f(x_0)| \leq |f(x_0) - f(x_n)| + |f(x_n) - f(y_n)|, \quad n = 1, 2, \cdots.$$

So, we have extended f to be defined on $\overline{K}$. Next we prove f is uniformly continuous on $\overline{K}$.

For any $\epsilon > 0$, since f is uniformly continuous on K, there is a $\delta > 0$ such that if $x, y \in K$ and $d(x, y) < \delta$ then $|f(x) - f(y)| < \epsilon$. For any $x, y \in \overline{K}$ and $d(x, y) < \delta/3$, by definition, there are $x', y' \in K$ such that

$$d(x, x') < \delta/3, \; |f(x) - f(x')| < \epsilon; \quad d(y, y') < \delta/3, \; |f(y) - f(y')| < \epsilon.$$

Therefore,

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) < \delta$$

and

$$|f(x) - f(y)| \leq |f(x) - f(x')| + |f(x') - f(y')| + |f(y') - f(y)| < \epsilon + \epsilon + \epsilon = 3\epsilon.$$

Therefore: f is uniformly continuous on $\bar{K}$. ∎

EXAMPLE 4 *Let $f(x)$ be a uniformly continuous function on $\mathbb{Q}$. Show that there is uniformly continuous function F on $\mathbb{R}$ s.t $F|_{\mathbb{Q}} = f$.*

Proof. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we have $\overline{\mathbb{Q}} = \mathbb{R}$. The construction of F will be obtained by repeating the argument of the proof of the previous theorem. ∎

1.3 Inverse function

Definition 1.5 *Inverse Function: Let $f : X \to Y$ be one -to -one and onto. Then we define an inverse function $f^{-1} : Y \to X$ as follow: $f^{-1}(y) = x$ if $f(x) = y$.*

THEOREM 1.6 *If (X, d_X) and (Y, d_Y) are two metric spaces, X is compact. If $f : X \to Y$ is one-to-one and onto and if f is continuous on X then $f^{-1} : Y \to X$ is also continuous on Y.*

Lemma 1.7 *If $f : X \to Y$ is continuous and $K \subset X$ is a compact subset, then $f(K)$ is compact in Y.*

Proof. For any open covering $\{V_\alpha : \alpha \in I\}$ for $f(K)$, we have that $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an open cover for K. Since K is compact, there is a finite subcover: $\{f^{-1}(V_{\alpha_j})\}_{j=1}^n$ with $K \subset \cup_{j=1}^n f^{-1}(V_{\alpha_n})$. Thus, $f(K) \subset \cup_{j=1}^n V_{\alpha_j}$. So, $f(K)$ is compact. ∎

Now we prove our theorem.

Proof. Let U be any open set in X. Then U^c is closed in X. Since X is compact, so U^c is compact in X. Thus, $f(U^c)$ is compact in Y. So $f(U^c)$ is closed in Y. Now, since f is $1-1$, $(f^{-1})^{-1}(U) = f(U) = Y \setminus f(U^c)$ is open, so $f^{-1} : Y \to X$ is continuous. ∎

1.4 Sequences of functions

Definition 1.8 *Let (X, d_X) and (Y, d_Y) be two metric spaces. Let f_n, f be functions from X to Y for $n = 1, 2, 3, \cdots$. Let $K \subset X$. Then*

(1) $f_n(x) \to f(x)$ pointwise on K as $n \to \infty$ if for any $x \in K$ (fixed), $\lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0$.

(2) $f_n(x) \to f(x)$ uniformly on K as $n \to \infty$ if for any $\epsilon > 0$, $\exists\ N$ such that if $n \geq N$, then $d_Y(f_n(x), f(x)) < \epsilon$ for all $x \in K$.

EXAMPLE 5 *Let $f_n(x) = x^n$. Then*

(1) $K = [0, 1)$, $f_n(x) = x^n \to 0$ pointwise on $[0, 1)$ as $n \to \infty$.

(2) $K = [0, 1]$. $f_n(x) \to f(x) = \{0$ if $x \in [0, 1)$ and 1 if $x = 1\}$ pointwise, on $K = [0, 1]$;

(3)$f_n(x) = x^n$ does not converge to 0 uniformly on $K = [0, 1)$.

Proof. (1) and (2) are easily seen. To prove (3). Let $\epsilon_0 = \frac{1}{2}$, for any N, then $x_N = \sqrt[N]{\frac{1}{2}} \in [0, 1)$, but $|f_N(x_N) - 0| = |(x_N)^N - 0| = |1/2 - 0| = 1/2 = \epsilon_0$. So, $f_n \to 0$ is not uniformly on $[0, 1)$ as $n \to \infty$.

THEOREM 1.9 *(X, d_X) and (Y, d_Y) are two metric spaces $K \subset X$ is a subset. If $f_n, : K \to Y$ are continuous on K. f is a function on K and $f_n \to f$ uniformly on K as $n \to \infty$, then f is continuous on K. i.e. the uniform limit of continuous functions is continuous.*

Proof. We need to prove f is continuous on K. For $x_0 \in K$, we will show f is cont. at x_0. For $\epsilon > 0$, we need to find $\delta > 0$ s.t. if $x \in K$, $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \epsilon$.

Since $f_n \to f$ uniformly on K, for the $\epsilon > 0$, there is N s.t. if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in K$. Since f_N is continuous at x_0 for the $\epsilon > 0$, there is $\delta_1 > 0$ s.t. if $x \in K$, $d_X(x, x_0) < \delta_1$, then $|f_N(x) - f_N(x_0)| < \epsilon$. Now, let $\delta = \delta_1$, when $x \in K$, $d_X(x, x_0) < \delta$. we have

$$\begin{aligned} |f(x) - f(x_0)| \quad & \leq |f(x) - f_n(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ & \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ & < \epsilon + \epsilon + \epsilon = 3\epsilon \end{aligned}$$

Thus, f is continuous at x_0, so f is continuous on K.

THEOREM 1.10 *Let $K \subset X$ be a compact subset of X. $f_n : K \to \mathbb{R}$ is continuous on K and $f_n(x) \geq f_{n+1}(x)$. If f is continuous on K and if $f_n(x) \to f(x)$ pointwise on K, then $f_n(x) \to f(x)$ uniformly on K.*

Proof. Without loss of generality (WLOG), we may assume $f = 0$, otherwise we use $g_n = f_n - f$ to replace f_n.

For any $\epsilon > 0$, let $K_n(\epsilon) = \{x \in K : f_n(x) \geq \epsilon\}$. Since f_n is continuous on K, we know $K_n(\epsilon)$ is closed. Since $K_n(\epsilon) \subset K$ and K is compact, therefore $K_n(\epsilon)$ is compact.

Since $f_n \to 0$ pointwise on K, we have $\cap_{n=1}^{\infty} K_n(\epsilon) = \emptyset$. We claim there is N s.t. $K_N(\epsilon) = \emptyset$. If not, $K_n(\epsilon) \neq \emptyset$ for all $n = 1, 2, ...$ Then $\cap_{n=1}^{\infty} K_n(\epsilon) \neq \emptyset$. This is s contradiction. Therefore, there is N such that $K_N(\epsilon) = \emptyset$. Therefore,$K_n(\epsilon) \subset K_N(\epsilon) = \emptyset$ when $n \geq N$. This means that $0 \leq f_n(x) < \epsilon$, $x \in K$ when $n \geq N$. So, $f_n(x) \to 0$ unif. on K. ∎

EXAMPLE 6 $p_0 = 0$, $p_{n+1}(x) = p_n(x) + \frac{x^2 - p_n(x)^2}{2}$, $x \in [-1, 1]$. *Show* $p_n(x) \to |x|$ *uniformly on* $[-1, 1]$.

Proof. Since $p_0(x) = 0$, $p_1(x) = 0 + \frac{x^2 - 0}{2} = \frac{x^2}{2} \geq 0$, $x \in [-1, 1]$.

We claim: $0 \leq p_n(x) \leq |x|$, $x \in [-1, 1]$, $n = 1, 2, ...$

We use induction to prove the claim.

When $n = 1$, the claim is true.

Assume the claim is true for n. We will prove it is true for $n + 1$.

Notice $(t - t^2/2)' = 1 - t \geq 0$ when $|t| \leq 1$, thus $t - t^2/2$ is increasoing on $|t| \leq 1$, we have $p_{n+1}(x) \leq |x| + \frac{x^2 - |x|^2}{2} \leq |x|$, and $p_{n+1}(x) = p_n(x) + \frac{x^2 - p_n(x)^2}{2} \geq 0$.

So the claim is true for $n+1$. Thus, by math induction, we have $0 \leq p_n(x) \leq |x|$, $x \in [-1, 1]$, $n = 0, 1, 2, ...$ and $p_{n+1}(x) = p_n(x) + \frac{x^2 - p_n(x)^2}{2} \geq p_n(x)$; $x \in [-1, 1]$ for all $n = 1, 2, 3, \cdots$.

Therefore $\lim_{n\to\infty} p_n(x) = f(x)$ exists in $\mathbb{R}$ for each $x \in [-1,1]$. Therefore, $f(x) = f(x) + \frac{x^2 - f(x)^2}{2}$. Thus, $f(x) = |x|$ on $[-1,1]$ which is continuous on $[-1,1]$. Since $p_n(x) \le p_{n+1}(x)$, $x \in K = [-1,1]$. Since $K = [-1,1]$ is compact, by previous theorem, $p_n(x) \to |x|$ uniformly on $[-1,1]$. □

1.5 Exercises

1. Assume that $m, n \ge 0$, $k > 0$ and $m + n > 2k$. Prove

$$\lim_{(x,y)\to(0,0)} \frac{x^m y^n}{x^{2k} + y^{2k}} = 0$$

2. Prove $\sqrt[3]{x}$ and $g(x) = \frac{x^2}{1+x^2}$ are uniformly continuous on $(0,\infty)$.

3. Let $f(x)$ be uniformly continuous on $\mathbb{R} \setminus \mathbb{Q}$, the set of irrational numbers. Prove there is a uniformly continuous function $F(x)$ on $\mathbb{R}$ so that $F(x) = f(x)$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$.

4. Prove $f(x) = x^3$ is not uniformly continuous on $(0,\infty)$.

5. Assume that $f : [0,1] \to [0,1]$ is monotone increasing. Prove that there is $x \in [0,1]$ so that $f(x) = x$.

6. Let X be a connected metric space. Let $f : X \to \mathbb{R}$ be a continuous function. If $c, d \in f(X)$ with $c < d$, then for any $s \in (c,d)$ there is $x_s \in X$ so that $f(x_s) = s$.

7. Prove that $f(x,y) = \sqrt{(1 + x^2 + y^2)}$ is uniformly continuous on $\mathbb{R}^2$.

8. Prove $f(x) = x \sin\Big(1/x\Big)$ is uniformly continuous on $(0,\infty)$.

9. Let f be continuous on (a,b) such that

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x) + f(y)}{2}, \quad x, y \in (a,b).$$

Prove that $f(x)$ is convex on (a,b).

10. Prove $f(x) = x^{10} - x^3 - 1$ has at least one zero on $(-1,1)$.