**Summer Jump-Start Program for Analysis, 2013**
Song-Ying Li

# 1 Lecture 10: Maxima and Minima

## 1.1 Taylor Theorem in $\mathbb{R}^n$

**THEOREM 1.1** *Let $U$ be an open convex set in $\mathbb{R}^n$. Let $f(x) \in C^{n+1}(U)$ and $x_0 \in U$. Then for any $x \in U$, there is $\xi \in [x_0, x]$ such that*

$$f(x) = \sum_{|\alpha|=0}^{n} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f(x_0)}{\partial x^\alpha} (x - x_0)^\alpha + \sum_{|\alpha|=n+1} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f(\xi)}{\partial x^\alpha} (x - x_0)^\alpha.$$

*Here:* $\alpha = (\alpha_1, ...., \alpha_n)$, $\alpha_j \in \mathrm{Z}_+$, $|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha! = \alpha_1!....\alpha_n!$.

**Proof.** Since $U$ is convex and $x_0, x \in U$, we have

$$[x_0, x] = \{tx + (1-t)x_0 : t \in [0, 1]\} \subset U.$$

We let $g(t) = f(tx + (1-t)x_0)$. Then, by the Taylor theorem of one variable, one has

$$f(x) = g(1) = \sum_{j=0}^{n} \frac{g^{(j)}(0)}{j!} (1-0)^j + \frac{g^{(n+1)}(\theta)}{(j+1)!} 1^{n+1}$$

for some $\theta \in (0, 1)$. Notice that

$$g(0) = f(x_0), \quad g'(0) = \frac{d}{dt} f(tx + (1-t)x_0)|_{t=0} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_0)(x_j - x_j^0)$$

and

$$\begin{aligned} g''(0) &= \frac{d}{dt} \left( \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} (tx + (1-t)x_0)(x_j - x_j^0) \right) |_t = 0 \\ &= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} (tx + (1-t)x_0)(x_i - x_i^0)(x_j - x_j^0) \text{ restricts } t = 0 \\ &= \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} (0)(x_i - x_i^0)(x_j - x_j^0) \end{aligned}$$

It is easy to show $\frac{1}{2!} g''(0) = \sum_{|\alpha|=2} \frac{\partial^2 f(x_0)}{\partial x^\alpha} \frac{1}{(\alpha!)} (x - x_0)^\alpha$. Similarly, one can prove $\frac{1}{j!} g^{(j)}(t) = \sum_{|\alpha|=j} \frac{1}{\alpha!} \frac{\partial^j f(tx_0 + (1-t)x)}{\partial x^\alpha} (x - x_0)^\alpha$ for $j = 1, 2, ...n + 1$. ∎

## 1.2 Extremal problems

Let $f$ be a continuous function in a domain $U$. We have the following definitions:

**Definition 1.2** *Let $x_0 \in U$. Then*

*a) We say that $x_0$ is a local maximizer for $f$ if there is a $\delta > 0$ such that*

$$f(x_0) \geq f(x), \quad x \in B(x_0, \delta)$$

*b) We say that $x_0$ is a local minimizer for $f$ if there is a $\delta > 0$ such that*

$$f(x_0) \leq f(x), \quad x \in B(x_0, \delta)$$

*c) We say that $x_0$ is a global maximizer for $f$ on $U$ if*

$$f(x_0) \geq f(x), \quad x \in U$$

*d) We say that $x_0$ is a global minimizer for $f$ on $U$ if*

$$f(x_0) \leq f(x), \quad x \in U$$

• **A major question is:**

How to find maximum (or maximizer) and minimum of $f$ in $U$ if they exist?

We start with the following proposition.

**Proposition 1.3** *Let $f \in C^1(U)$. Then if $x_0 \in U$ is a local maximizer or a local minimizer of $f$ in $U$, then $\nabla f(x_0) = 0$.*

**Definition 1.4** *A point $x_0 \in U$ is called a critical point of $f$ in $U$ if either $\nabla f(x_0) = 0$ or $\nabla f(x_0)$ does not exist. $x_0$ .*

**Question.** If $x_0 \in$ is a critical point for $f$ in $U$. How to test if $x_0$ is a local maximizer or minimizer or a saddle point?

• Here we will introduce a test called the 2nd derivative test.

**THEOREM 1.5** *Let $U$ be an open set in $\mathbb{R}^n$ and $x_0 \in U$. For $f \in C^2(U)$, let*

$$D^2 f(x_0) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(x_0) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x_0) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x_0) \end{bmatrix}.$$

*Then the following statements hold:*

*(a) If $D^2 f(x_0)$ is positive definite, then $x_0$ is local minimizer;*
*(b) If $D^2 f(x_0)$ is negative definite, then $x_0$ is local maximizer;*
*(c) iI $D^2 f(x_0)$ is indefinite, then $x_0$ is a saddle point.*

**EXAMPLE 1** *Let* $f(x_1, x_2) = x_1^2 + x_2^2$*. Then* $(0,0)$ *is a critical point.*

$$D^2 f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

*is positive definite, so* $(0,0)$ *is a local minimizer.*

**EXAMPLE 2** *Let* $g(x_1, x_2) = -(x_1^2 + x_2^2)$*. Then* $(0,0)$ *is a critical point.*

$$D^2 g(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

*is negative definite, so* $(0,0)$ *is a local maximizer.*

**EXAMPLE 3** *Let* $h(x_1, x_2) = x_1^2 - x_2^2$*. Then* $(0,0)$ *is a critical point.*

$$D^2 h(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

*is indefinite, so* $(0,0)$ *is a saddle point.*

**EXAMPLE 4** *Let* $f(x,y) = x^4 + y^4$*. Then* $(0,0)$ *is a critical point and*

$$D^2 f(x,y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix}, \quad D^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*So, the 2nd derivative test fails. But* $(0,0)$ *is the global minimum for* $f$ *in* $\mathbb{R}^2$*.*

In general, we consider the convex function.

**Definition 1.6** *Let* $U$ *be a convex set in* $\mathbb{R}^n$*. A function* $f$ *on* $U$ *is said to be convex on* $U$ *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)(f(y)) \quad \text{for all } x, y \in U,\ \lambda \in [0,1].$$

**THEOREM 1.7** *Let* $U$ *be a convex set in* $\mathbb{R}^n$*. Then* $f \in C^2(U)$ *is convex on* $U$ *if and only* $D^2 f(x)$ *is positive semidefinite on* $U$ *(i.e.* $\sum_{i,j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} a_i a_j \geq 0$ *for all* $a$ *in*$\mathbb{R}^n$*,* $x \in U$*).*

**Proof.** It is easy to see that $f$ is convex in $U$ if and only if $g(t) = f(tx+(1-t)y)$ is convex on $[0,1]$ for all $x, y \in U$ if only if $g''(t) \geq 0$ on $[0,1]$ and $x, y \in U$. Notice that

$$g''(t) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(tx + (1-t)y)(x_i - y_i)(x_j - y_j)$$

one can see that $f$ is convex if and only if

$$\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x)(x_i - y_i)(x_j - y_j) \geq 0, \quad y \in U, x \in U.$$

This gives the proof of the theorem. ∎

**THEOREM 1.8** *If $U$ is convex and $f \in C^2(U)$ is convex on $U$, then every critical point of $f$ is a global minimizer of $f$ in $U$.*

**Proof.** Let $x_0$ be any critical point of $f$ in $U$. Then $\nabla f(x_0) = 0$. By the Taylor Theorem and $f$ being convex,

$$\begin{aligned} f(x) &= f(x_0) + \nabla f(x_0) \cdot (x - x_0) + \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} (\theta x_0 + (1-\theta)x)(x_i - x_i^0)(x_j - x_j^0) \\ &\geq f(x_0). \end{aligned}$$

Therefore, $f(x_0)$ is global minimum of $f$ in $U$. □

**EXAMPLE 5** *Let $f(x,y) = x^4 + y^4 - 32x - 2y^2$. Find all global minimizers of $f$ in $\mathbb{R}^2$.*

**Solution.** Since

$$\begin{cases} \frac{\partial f}{\partial x} = 4x^3 - 32 = 0 \\ \frac{\partial f}{\partial y} = 4y^3 - 4y = 0 \end{cases}$$

has three solutions: $(2,0)$, $(2,1)$ and $(2,-1)$. Which are critical points of $f$. Notice that

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 - 4 \end{bmatrix}$$

Then

$$H(f)(2,0) = \begin{bmatrix} 48 & 0 \\ 0 & -4 \end{bmatrix}$$

is indefinite, so $(2,0)$ is a saddle point of $f$; and $H(f)(2,\pm 1)$ are positive definite. So $(2,\pm 1)$ both are local minimizers and $F(2,\pm 1) = -48 - 1 = -49$. Since $f(x,y) \to +\infty$ as $x^2 + y^2 \to +\infty$. Therefore, the both $(2,\pm 1)$ are global minimizers for $f$ on $\mathbb{R}^2$.

## 1.3 LaGrange Multipliers

We study the maximizing or minimizing problem with constraints.

$$\begin{cases} \text{Maximize (or minimize)} : f(x,y,z). \\ \text{Subect to: } g(x,y,z) = c. \end{cases}$$

Since the maximizer or minimizer must take place at $x$ where:

$$\nabla f(x,y,z) \,||\, \nabla g(x,y,z) \quad \text{and } g(x,y,z) = c.$$

In other words, we solve the critical points from the system of equations:

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = c. \end{cases}$$

**EXAMPLE 6** *Find maximum and minimum of* $f(x,y,z) = x + y$ *on* $S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$.

**Solution.** We want to solve

$$\begin{cases} \text{Maximize (or minimize):} f(x,y,z) = x + y; \\ \text{subject to: } g = x^2 + y^2 + z^2 = 1. \end{cases}$$

We solve for $(x,y,z)$ from:

$$\begin{aligned} \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) \\ g(x,y,z) &= c. \end{aligned}$$

Which is:

$$\begin{aligned} 1 &= 2\lambda x \\ 1 &= 2\lambda y \\ 0 &= 2\lambda z \\ 1 &= x^2 + y^2 + z^2. \end{aligned}$$

This implies, $z = 0$ and $x = y = 1/(2\lambda)$. Thus

$$1 = (1/2\lambda)^2 + (1/(2\lambda))^2$$

So, $2\lambda^2 = 1$. Therefore, $\lambda = \pm\sqrt{1/2}$. Therefore, we have solutions:

$$(1/\sqrt{2}, 1/\sqrt{2}, 0), (-1/\sqrt{2}, -1/\sqrt{2}, 0)$$

Maximum for the problem is: $f(1/\sqrt{2}, 1/\sqrt{2}) = \sqrt{2}$, and minimum for the problem is: $f(-1/\sqrt{2}, -1/\sqrt{2}) = -\sqrt{2}$.

## 1.4 Answer for some Exercise or test problems

**EXAMPLE 7** *Given an example of continuous function* $f$ *on* $\mathbb{R}^2$ *such that* $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ *exist on* $\mathbb{R}^2$*, but* $f$ *is not differentiable at* $(0,0)$.

**Solution.** Let

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$f(x,y)$ is continuous and diff. on $\mathbb{R}^2 \backslash \{(0,0)\}$. So $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist there. For $(0,0)$, since $\frac{-1}{2}|x| \leq \frac{x^2 y}{x_2+y^2} \leq \frac{1}{2}|x|$ since $|xy| \leq \frac{1}{2}(x^2+y^2)$, by the squeeze limit theorem,

$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x_2+y^2} = 0 = f(0,0)$. So $f$ is continuous at $(0,0)$ and $f \in C(\mathbb{R}^2)$. Thus,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{0}{x} = 0$$
$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(y,0)-f(0,0)}{y} = \lim_{y\to 0} 0/y = 0$$

So $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ exist on $\mathbb{R}^2$.

Next we show $f$ is not differentiable at $(0,0)$. Otherwise,

$$f'(0,0) = \nabla f(0,0) = [0\ 0]$$

and

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - \frac{\partial f}{\partial x}(0,0)x - \frac{\partial f}{\partial y}(0,0)y|}{\sqrt{x^2+y^2}} = 0.$$

But,

$$\text{LHS} = \lim_{x\to 0, y\to 0} \frac{\frac{|x^2y|}{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{x=y} \frac{x^2|x|}{(x^2+x^2)^{3/2}} = \lim_{x\to 0} \frac{|x|^3}{\sqrt{8}|x|^3} = \frac{1}{2\sqrt{2}} \neq 0.$$

This is a contradiction. ∎

**EXAMPLE 8** *Suppose $f, f', f''$ are continuous and bounded on $(-\infty, \infty)$, $f(0) = f'(0) = 0$. Discuss the uniform convergence for series $\sum_{n=1}^{\infty} f(\frac{x}{n})$.*

**Solution** By Taylor theorem:

$$f(x) = f(0) = f'(0)x + \frac{f''(x_0) + \theta(x-x_0)}{2!}(x-x_0)^2 = \frac{1}{2!} f''(x_0 + \theta(x-x_0))x^2.$$

$|f(x/n)| \leq M_2(1/2)(x^2/n^2)$. Therefore, $\sum_{n=1}^{\infty} f(x/n^2)$ converges uniformly, absolutely on $[-M, M]$ for any $M > 0$ by Weierstrass M-test and $\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$.

Now the question is: Can $\sum_{n=1}^{\infty} f(x/n)$ converge uniformly on $(-\infty, \infty)$?

Answer: No. Let

$$f(x) = \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$$

Then

$$f'(x) = \frac{2x}{(1+x^2)^2}, \quad f''(x) = \frac{2}{(1+x^2)^2} - \frac{8x^2}{(1+x^2)^3}.$$

Thus $f, f', f''$ are continuous and bounded on $\mathbb{R}$.

$$\sum_{n=1}^{\infty} f(x/n) = \sum_{n=1}^{\infty} \frac{\frac{x^2}{n^2}}{1+\frac{x^2}{n^2}} = \sum_{n=1}^{\infty} \frac{x^2}{n^2+x^2}$$

does not converge uniformly on $(-\infty,\infty)$ because $\lim_{n\to\infty}\frac{x^2}{n^2+x^2}=0$ is not uniformly on $\mathbb{R}$ since for any $N$, let $x_N=N\in(-\infty,\infty)$ then $\frac{x_N^2}{1+x_N^2}=\frac{N^2}{2N^2}=\frac{1}{2}$.

**EXAMPLE 9** *Let $f$ be differentiable on $[a,b]$ such that $f(0)=0$ and $|f'(x)|\le A|f(x)|$ on $[a,b]$ for some positive uumber $A$. Prove $f\equiv 0$ on $[a,b]$.*

**Proof.** By Fundamental Thmeorem of Calculus:

$$f(x)=f(0)+\int_0^x f'(t)dt=\int_a^x f'(t)dt.$$

Let $x_1\in[0,\frac{1}{2A}]$ be a maximum point for $f$ on $[0,\frac{1}{2A}]$. Then

$$\begin{aligned}|f(x_1)| &= \Big|\int_0^{x_1} f'(t)dt\Big| \\ &\le \int_0^{x_1}|f'(t)dt \le \int_0^{x_1} A|f(t)|dt \\ &\le A|f(x_1)|\int_0^{x_1} dt \le A|f(x_1)|x_1 \le (1/2)|f(x_1)| \le 0\end{aligned}$$

Therefore, $|f(x_1)|=0$ and $f(x)=0$ on $[a,a+\frac{1}{2A}]$.

Similarly, we can prove $f=0$ on $[a+\frac{1}{2A},a+\frac{i+1}{2A}]$, $i=0,1,2,...$ when $\frac{j+1}{A}>b-a$, so we have $f=0$ on $[a,b]$. ∎

**EXAMPLE 10** *Let $f\in C^3([-1,1])$ be such that $f(-1)=0$, $f(0)=0$, $f(1)=1$, $f'(0)=0$. Prove there is a $x_0\in(-1,1)$ such that. $f'''(x_0)\ge 3$.*

**Proof.** Since

$$f(1)=f(0)+f'(0)+(1/2)f''(0)+\frac{f^{'''}(\xi_1)}{3!}\text{ for }\xi\in(0,1),\quad 1=(1/2)f''(0)+\frac{f^{'''}(\xi_1)}{3!}.$$

Similarly,

$$0=f(-1)=\frac{1}{2}f''(0)+\frac{f^{'''}(\xi_2)}{3!}(-1)^3=\frac{1}{2}f''(0)-\frac{f^{'''}(\xi_2)}{3!}.$$

Thus

$$1=\frac{f^{'''}(\xi_1)}{3!}+\frac{f^{'''}(\xi_2)}{3!}=\frac{1}{6}(f''(\xi_1)+f'''(\xi_2)).$$

If $f'''(\xi_1)\ge f'''(\xi_0)$, then $\frac{2f'''(\xi_1)}{3!}\ge 1$, $f'''(\xi_1)\ge 3$, otherwise, $f'''(\xi_2)\ge 3$. ∎

## 1.5 Exercise

1. Let $D$ be a convex open set in $\mathbb{R}^n$. Prove any convex harmonic functions on $D$ must be linear functions.

2. Use the method of Lagrange Multiplier to solve

$$\begin{cases} \text{Minimize:} & f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 - x_2 + 3x_3 \\ \text{Subject to:} & 2x_3 = x_1^2 + x_2^2 \end{cases}$$

3. Classify all critical points of $f$:

$$f(x_1, x_2, x_3) = x_1^2 + x_2^3 - 3x_1x_2 + x_3^2$$

4. Given $n$ distinct points $(x_1, y_1), \cdots, (x_n, y_n)$ in $\mathbb{R}^2$. Find the equation of a line $y = ax + b$ such that $\sum_{j=1}^{n}(ax_j + b - y_j)^2$ is minimum. (Such line is called regression line for those $n$-points.)

5. Let $P = (1, -10, 20)$. Find the distance from $P$ to the unit sphere in $\mathbb{R}^3$.

6. Let $U$ be a convex set in $\mathbb{R}^n$, $u$ is a convex function on $U$ and $f$ is convex increasing function on $\mathbb{R}$. Then $f \circ u$ is convex on $U$.

7. Where is $f(x) =: \ln(x_1^2 + \cdots + x_n^2)$ is convex ?