Summer Jump-Start Program for Analysis, 2013
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1 Lecture 11: Riemann Integrals

1.1 Definition of Riemann integrals

Let $f(x)$ be a bounded function on a bounded closed integral $[a, b]$. First, we consider a partition P for $[a, b]$ as follows:

$$P : a = x_0 < x_1 < .... < x_n = b.$$

with the norm of the partition P defined as

$$||P|| = \max\{\Delta x_i, i = 1, 2, ..., n\} \quad \text{and} \quad \Delta x_i = x_i - x_{i-1}.$$

Let

$$m_i(f) = \inf\{f(x) : x_{i-1} \le x \le x_i\} \quad \text{and} \quad M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}.$$

be the infimum and supremum of f on the subinterval $[x_{i-1}, x_i]$. Then we define the lower partial sum and the upper partial sum of f with respect to the partition P as follows:

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i; \quad \text{and } U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

It is easy to see that $L(f, P) \le U(f, P)$.

REMARK 1 *In fact, for any two partitions P_1 and P_2 for $[a, b]$, we always have $L(f, P_1) \le U(f, P_2)$.*

Proof. Let $P_1 : a = x_0 < x_1 < ... < x_n = b$ and $P_2 : a = y_0 < y_1 < .... < y_m = b$. We define a refinement for P_1 and P_2 which is partition P for $[a, b]$ with end points of subintervals are $x_0, ..., x_n$ and $y_0, ..., y_m$. We denote $P = P_1 \cup P_2$. Then

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2).$$

We define the Riemann lower and upper integrals of f on $[a, b]$ as follows:

$$L(f) = \lim_{||P|| \to 0} L(f, P) \quad \text{and} \quad U(F) = \lim_{||P|| \to 0} U(f, P).$$

The both are always exist.

Definition 1.1 *Let f be a bounded function on $[a, b]$. Then we say that f is (Riemann) integrable on $[a, b]$ if $L(f) = U(f)$.*

1.2 Tests for Riemann-integrability

THEOREM 1.2 *Let f be a bounded function on $[a,b]$. Then f is integrable on $[a,b]$ if and only if for any $\epsilon > 0$, there is a partition P for $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.*

Proof. "$\Rightarrow$ " Assume f R-integrable. Then, $L(f) = U(f)$. For $\epsilon > 0$, there is P_1 and P_2 such that $-L(f,P_1) + L(f) < \epsilon/2$ and $U(f,P_2) - U(f) < \epsilon/2$ Since $L(f) = U(f)$ and $P = P_1 \cup P_2$, we have $U(f,P) - L(f,P) < \epsilon$.

" $\Leftarrow$ " Since $L(f) \geq L(f,P)$, $U(f) \leq U(f,P)$. If for any $\epsilon > 0$, there is P such that $U(f,P) - L(f,P) < \epsilon$, then $U(f) - L(f) < \epsilon$. But, $U(f) - L(f) \geq 0$ is a fixed number and $U(f) - L(f) < \epsilon$ for any $\epsilon > 0$, then $U(f) - L(f) = 0$. Since $U(f) = L(f)$, f is integrable.

EXAMPLE 1 *If f is continuous on $[a,b]$, then f is integrable.*

Proof. For any $\epsilon > 0$, since f is continuous on $[a,b]$, it is uniformly continuous on $[a,b]$. So, there is $\delta > 0$ such that if $x_1, x_2 \in [a,b]$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon/(b-a)$. Now, choose n such that $\frac{b-a}{n} < \delta$. Let $P : x_0 = a < x_1 < .... < x_n = b$, where $x_j = x_{j-1} + \frac{b-a}{n}$. Then, $M_j - m_j < \epsilon/(b-a)$. $U(f,P) - L(f,P) \leq \sum_{j=1}^{n} (M_j - m_j)\Delta x_j < \frac{\epsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \epsilon$

EXAMPLE 2 *Let*

$$D(x) = f((x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ irrational} \end{cases}$$

$D(x)$ is not integrable on any interval $[a,b]$. In fact, $U(f) = (b-a)$, $L(f) = 0$.

Note: The Dirichlet function $D(x)$ is the typical examples of f which is not integrable on $[a,b]$.

EXAMPLE 3 *Let*

$$R(x) = \begin{cases} 0 & x \text{ irrational} \\ 1/n & x = m/n, (m,n) = 1, x \in [0,1] \end{cases}$$

Then $R(x)$ is integrable on $[0,1]$.

Exercise: Prove it on your own!

EXAMPLE 4 *If $f(x)$ is a monotone increasing function on $[a,b]$, then it is integrable.*

Proof. Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition for $[a,b]$. Since f is increasing, we have

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\} = f(x_{i-1})$$

and

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} = f(x_i).$$

Since

$$M_1 - m_1 + M_2 - m_2 + \cdots + M_n - m_n \le M_n - m_1 \le f(b) - f(a),$$

we have

$$U(f,P) - L(f,P) = \sum (M_i - m_i)\Delta x_i \le ||P|| \sum_{i=1}^{n} (M_i - m_i) \le (f(b) - f(a))||P||$$

For $\epsilon > 0$, and for any partition P for $[a,b]$ with $||P|| < \frac{\epsilon}{f(b)-f(a)}$, we have $U(f,P) - L(f,P) < (f(b) - f(a))\frac{\epsilon}{f(b)-f(a)} = \epsilon$. So, f is integrable on $[a,b]$.

• Let f be bounded on $[a,b]$. Let $D_s(f)$ be the set of all points $x_0 \in [a,b]$ with f is discontinuous at x_0.

THEOREM 1.3 *Let f be bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if $m(D_s) = 0$ (D_s has measure 0).*

• We say that D_s has measure 0 if: For $\epsilon > 0$, there is a sequence of intervals $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that $D_s \subset \bigcup_{j=1}^{\infty}(a_j, b_j)$ and $\sum_{j=1}^{\infty}(b_j - a_j) < \epsilon_0$.

1.3 Improper Integrals

THEOREM 1.4 *Let f be bounded on $[a,b]$. Then f is integrable on $[a,b]$ if and only if f^+ and f^- are integrable on $[a,b]$,where*

$$f^+(x) = \begin{cases} f(x), & f(x) \ge 0 \\ 0, & f(x) < 0 \end{cases} \quad \textit{and} \quad f^-(x) = \begin{cases} -f(x), & f(x) < 0 \\ 0, & f(x) \ge 0 \end{cases}$$

Corollary 1.5 *If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$. Converse is not true.*

EXAMPLE 5

$$f(x) = \begin{cases} 1 & \textit{if } x \textit{ is rational} \\ -1 & \textit{if } x \textit{ is irrational} \end{cases}$$

f is not integrable on $[a,b]$ since $L(f,P) = -(b-a)$ and $U(f,p) = (b-a)$, but $|f(x)| = 1$ is integrable on $[a,b]$.

Next we consider the integrality for some unbounded function on $[a,b]$.

Definition 1.6 *Let $f(x)$ be a function on (a,b) (usually f is unbounded). Assume for any $a < c < d < b$, we have $f(x)$ is integrable on $[c,d]$. We say that f is integrable on (a,b) If*

$$\int_a^b f(x)dx =: \lim_{c \to a, d \to b} \int_c^d f(x)dx \quad \textit{(exists)}$$

Proposition 1.7 *If f is a function on (a, b) and f is integral on $[c_1, c_2]$ for any $a < c_1 < c_2 < b$ and if $|f|$ is integrable on (a, b), then f is integrable on (a, b). The converse is not true.*

EXAMPLE 6 *Let $f(x) = \frac{1}{x^\alpha}$ on $(0, 1]$. Then*
(i) f is integrable on $(0, 1]$ when $\alpha < 1$;
(ii) f is not integrable on $(0, 1]$ when $\alpha \geq 1$.

Proof. Since

$$\int_\epsilon^1 f(x)dx = \int_\epsilon^1 \frac{1}{x^\alpha}dx = \frac{1}{1-\alpha}\bigg|_\epsilon^1 = \frac{1}{\alpha-1}\epsilon^{1-\alpha} - \frac{1}{\alpha-1} \ (\alpha \neq 1).$$

we have

$$\lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x^\alpha}dx = -\frac{1}{\alpha-1} + \lim_{\epsilon \to 0^+} \frac{1}{\alpha-1}\epsilon^{1-\alpha} = \begin{cases} < \infty, \text{if } \alpha < 1; \\ +\infty, \ \text{if } \alpha > 1 \end{cases}$$

and

$$\int_0^1 \frac{1}{x}dx = +\infty.$$

THEOREM 1.8 *(Comparison test) If $f(x)$ and $g(x)$ are continuous on (a, b) and $|g(x)|$ is integrable on (a, b) and $|f(x)| \leq |g(x)|$, $x \in [a, b]$, then $|f(x)|$ is integrable on (a, b) and so is $f(x)$.*

EXAMPLE 7 *Prove or disprove: $\int_0^1 \frac{1}{x^{1/2}} \sin(1/x)dx$ converges.*

Proof. Converges! Since $\frac{1}{x^{1/2}} \sin(1/x)$ is continuous on $(0, 1]$,

$$\left|\frac{1}{x^{1/2}} \sin(1/x)\right| \leq \frac{1}{x^{1/2}}, x \in (0, 1],$$

and we know that $\int_0^1 \frac{1}{x^{1/2}} < +\infty$. By comparison test, we have $\int_0^1 \frac{1}{x^{1/2}} \sin(1/x)dx$ converges.

EXAMPLE 8 *Determine if $\int_0^1 \frac{\sin(x)}{x^{3/2}}dx$ converges.*

Solution. Since

$$\sin(x) \leq x, \text{ for } x \in [0, \pi/2], \quad 0 \leq \frac{\sin(x)}{x^{3/2}} \leq \frac{x}{x^{3/2}} = \frac{1}{x^{1/2}}$$

and $\int_0^1 \frac{1}{x^{1/2}} < +\infty$. By comparison test, $\int_0^1 \frac{\sin(x)}{x^{3/2}}dx$ converges.

EXAMPLE 9 $\int_0^1 \frac{\cos(x)}{x^{3/2}}dx$.

Solution. Diverges. Since

$$\frac{\cos x}{x^{3/2}} \geq \frac{\cos 1}{x^{3/2}}, \quad x \in [0,1].$$

We know $\int_0^1 \frac{\cos 1}{x^{3/2}} dx = +\infty$. By comparison test, $\int_0^1 \frac{\cos x}{x^{3/2}} dx = +\infty$.

Definition 1.9 *(i)* $\int_a^b f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx$.
(ii) $\int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$.
(iii)$\int_{\infty}^{\infty} f(x)dx = \lim_{a \to -\infty, b \to \infty} \int_a^b f(x)dx$.

EXAMPLE 10 $f(x) = \frac{1}{x^\alpha}$, $x \in [1, \infty)$.

$$\int_1^\infty \frac{1}{x^\alpha} dx = \begin{cases} \textit{finite} & \alpha > 1 \\ +\infty & \alpha \leq 1 \end{cases}$$

EXAMPLE 11 *Determine if* $\int_0^\infty \frac{\sin(x)}{\ln(x+2)} dx$ *converges.*

Solution

$$\begin{aligned} \int_0^\infty \frac{\sin(x)}{\ln(x+2)} dx &= -\frac{\cos(x)}{\ln(x+2)} \Big|_{x=0}^\infty + \int_0^\infty \left(\frac{1}{\ln(x+2)} \right)' \cos(x) dx \\ &= \frac{1}{\ln 2} + \int_0^\infty \frac{-1}{(\ln(x+1))^2} \frac{1}{x+2} \cos x dx \end{aligned}$$

We know that if $p > 1$ then

$$\begin{aligned} \int_2^\infty \frac{1}{x} \frac{1}{(\ln x)^p} dx &= \int_2^\infty \frac{1}{(\ln x)^p} d \ln x \\ &= -\frac{1}{p-1} (\ln x)^{1-p} \Big|_{x=2}^\infty \\ &= \frac{1}{p-2} (\ln 2)^{1-p} \\ &< +\infty. \end{aligned}$$

Since $\frac{-1}{(\ln(x+2))^2} \frac{1}{x+2} \cos(x)$ is continuous on $(0, \infty)$ and

$$\begin{aligned} &\int_0^\infty \left| \frac{-1}{(\ln(x+2)^2} \frac{1}{x+2} \cos(x) \right| dx \\ \leq & \int_0^\infty \frac{1}{x+2} \frac{1}{\ln(x+2)^2} dx \\ = & -(\ln(x+2))^{-1} \Big|_{x=0}^\infty \\ = & (\ln 2)^{-1} < +\infty \end{aligned}$$

So $\int_0^\infty \frac{-1}{(\ln(x+2))^2}\frac{1}{x+2}\cos x dx$ converges and so does $\int_0^\infty \frac{\sin(x)}{\ln(x+2)}$.

• Important techniques for integrations:

(1) Integration by parts: $f(x)dx = f(x)g(x)|_a^b - \int_A^b f'(x)g(x)dx$.
(2) Substituion $\Phi(t) : [a,b] \to [c,d]$ is onto, then, by letting $y = \Phi(x)$,

$$\int_a^b f(\Phi(x))\Phi'(x)dx = \int_c^d f(y)dy.$$

THEOREM 1.10 *(Newton-Leibniz Theorem) If $f(x)$ is integrable on $[a,b]$ and $F(x) = \int_a^x f(t)dt$. Then $F'(x_0) = f(x_0)$ when f is continuous at x_0.*

Note: $F = \int_a^x f(t)dt$ is called an anti-derivative of f on $[a,b]$.

EXAMPLE 12 *Let $F(x) = \int_{x-\cos(x)}^{x^2-\sin(x)} e^{t^2}(t+2)dt$. Find $F'(x)$.*

Solution Since

$$\begin{aligned} F(x) &= \int_0^{x^2-\sin(x)} e^{t^2}(t+2_dt + \int_{x-\cos(x)}^0 e^{t^2}(t+2)dt \\ &= \int_0^{x^2-\sin(x)} e^{t^2}(t+2)dt - \int_0^{x-\cos(x)} e^{t^2}(t+2)dt, \end{aligned}$$

we have

$$\begin{aligned} F'(x) &= \left[\int_0^{x^2-\sin(x)} e^{t^2}(t+2)dt\right]' - \left[\int_0^{x-\cos(x)} e^{t^2}(t+2)dt\right]' \\ &= e^{(x^2-\sin(x))^2}(x^2-\sin(x)+2)(x^2-\sin x)' - e^{(x-\cos(x))^2}(x-\cos x+2)(x-\cos x)' \\ &= e^{(x^2-\sin(x))^2}(x^2-\sin(x)+2)(2x-\cos x) - e^{(x-\cos(x))^2}(x-\cos x+2)(1+\sin x). \end{aligned}$$

1.4 Important inequalities

THEOREM 1.11 *Let f and g are (bounded) integrable on $[a,b]$. Then*
(i) fg is integrable on $[a,b]$
(ii) (Cauchy-Schwarz inequality:)

$$\left(\int_a^b f(x)f(x)dx\right)^2 \le \int_a^b f(x)^2dx \int_a^b g(x)^2dx.$$

Proof. (i) (Exercise for integrability). (ii) We consider:

$$\begin{aligned} 0 &\le \int_a^b (f(x)+\lambda g(x))^2 dx \\ &= \int_a^b f(x)dx + 2(\int_a^b f(x)g(x)dx)\lambda + (\int_a^b g(x)^2dx)\lambda^2, \quad \lambda \in (-\infty,\infty). \end{aligned}$$

Therefore,

$$[2\int_a^b f(x)g(x)dx]^2 - 4[\int_a^b g(x)^2dx][\int_a^b f(x)dx] \le 0$$

This implies

$$\Big(\int_a^b f(x)g(x)dx\Big)^2 \le \int_a^b f(x)^2dx \int_a^b g(x)^2dx$$

This completes the proof of Part (ii). □

THEOREM 1.12 *(Hölder's inequality): Assume f, g are (bounded) integrable on $[a,b]$. If $1 \le p,\ q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then fg is integrable on $[a,b]$ and*

$$\Big|\int_a^b f(x)g(x)dx\Big| \le (\int_a^b |f(x)|^p dx)^{1/p}(\int_a^b |g(x)|^q)^{1/q}.$$

Recall: (i) $xy \le \frac{1}{2}(x^2+y^2)$.

(ii) if $\frac{1}{p} + \frac{1}{q} = 1$, then $xy \le \frac{x^p}{p} + \frac{y^q}{q}$ for $x, y > 0$.

[Since $xy = e^{\frac{1}{p}\ln x^p + \frac{1}{q}\ln y^q} \le \frac{1}{p}e^{\ln x^p} + \frac{1}{q}e^{\ln x^q}$ (e^x is convex) $= \frac{1}{p}x^p + \frac{1}{q}y^q$.]

Proof. WLOG, we assume $f(x) \ge 0$, $g(x) \ge 0$. We divide it into two cases.

Case 1: Assume that $\int_a^b f(x)^p dx = 1$ and $\int_a^b g(x)^q dx = 1$. We need to prove: $\int_a^b f(x)g(x)dx \le 1$. Since

$$\begin{aligned}\int_a^b f(x)g(x)dx &\le \int_a^b \frac{1}{p}f(x)^p + \frac{1}{q}g(x)^q dx \\ &= \frac{1}{p}\int_a^b f(x)^p dx + \frac{1}{q}\int_a^b g(x)^q dx \\ &= \frac{1}{p}\cdot 1 + \frac{1}{q}\cdot 1 = 1.\end{aligned}$$

Case 2: General case: Let

$$F(x) = \frac{f(x)}{(\int_a^b f(x)^p dx)^{1/p}}, \quad G(x) = \frac{g(x)}{(\int_a^b g(x)^q dx)^{1/q}}.$$

Then $\displaystyle\int_a^b F(x)^p dx = \int_a^b \frac{1}{\int_a^b f(x)^p} dx \int_a^b f(x)^p dx = 1$ and $\int_a^b G(x)^q dx = 1$.

By Case 1, we have $\int_a^b F(x)G(x)dx \le 1$, This implies that

$$\int_a^b f(x)g(x)dx \le \Big(\int_a^b f(x)^p dx\Big)^{1/p}\Big(\int_a^b g(x)^q dx\Big)^{1/q}.$$

1.5 Exercise

1. Let $R(x)$ be the Riemann function defined as $R(x) = 0$ when x is irrational or 0, and $R(x) = 1/n$ if $x = m/n$ with $(m, n) = 1$, where n is positive integer. Prove $R(x)$ is integrable on $[0, 1]$ and find $\int_0^1 R(x)dx$.

2. Assume that $f(x)$ is a non-negative continuous function on $[a, b]$. If $\int_a^b f(x)dx = 0$, prove $f(x) = 0$ on $[a, b]$.

3. Assume that $f(x)$ is a real valued function on $[0, 1]$. Answer the following questions

(i) If $f(x)^2$ is integrable on $[0, 1]$, is $f(x)$ integrable on $[0, 1]$?
(ii) If $f(x)^3$ is integrable on $[0, 1]$, is $f(x)$ integrable on $[0, 1]$?
(iii) If f is integrable and bounded, is $f(x)^2$ integrable on $[0, 1]$?

4. Let $f(x)$ and $g(x)$ be two integrable functions on $[a, b]$ (both functions are bounded on $[a, b]$). Then one has the following Hölder inequality:

$$\Big|\int_a^b f(x)g(x)dx\Big| \le \Big(\int_a^b |f(x)|^p dx\Big)^{1/p}\Big(\int_a^b |f(x)|^{p'} dx\Big)^{1/p'}$$

where $1 \le p, p' < \infty$ and $1/p + 1/p' = 1$.

5. Prove

$$f(x) = \frac{\sin(x)}{\ln(2 + x)}$$

is integrable on $[0, \infty)$.

6. Prove

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(nx)}{1 + x} dx = 0$$

7. Prove or disprove the following improper integrals converge:

(a) $\int_0^\infty \frac{\sin t}{t^{3/2}} dt$, (b) $\int_0^\infty \frac{\cos t}{t} dt$ (c)$\int_0^1 \sin(\frac{1}{t})dt$, (d) $\int_0^1 \frac{1}{t}\sin(\frac{1}{t})dt$.

8. Define

$$f(x) = \int_x^{x+1} \sin(t^2)dt$$

(a) Prove $|f(x)| < 1/x$ if $x > 0$
(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x + 1)^2] + r(x)$$

where $|r(x)| < c/x$ and c is a constant.

(c) Find the upper and lower limits of $xf(x)$ as $x \to \infty$.
(d) Does $\int_0^\infty \sin(t^2)dt$ converge ?

9. Mean Value Theorem for integral: Let $f(x)$ is continuous on $[a, b]$, and $g(x)$ is non-negative and integrable on $[a, b]$. Prove that there is $x_0 \in (a, b)$ so that

$$\int_a^b f(x)g(x)dx = f(x_0) \int_a^b g(x)dx$$

19. Suppose f is real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f(x)^2 dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -1/2$$

and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f(x)^2 dx > 1/4.$$

11. Let α be a fixed increasing function on $[a, b]$. For integral function u on $[a, b]$, define

$$\|u\|_2^2 = \int_a^b |u(x)|^2 dx.$$

Suppose that f, g and h are Riemann-Stieltjes integrable with respect to α, prove the following triangle inequality:

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$