Summer Jump-Start Program for Analysis, 2013
Song-Ying Li

1 Lecture 12: Multiple integrals

1.1 Definition of Multiple integrals

Given a cell $E = [a_1, b_1] \times ... \times [a_n, b_n]$ in $\mathbb{R}^n$. A partition P for E is:

$$P: \quad a_1 = x_{10} < ... < x_{1m_1} = b_1$$
$$\vdots$$
$$a_n = x_{n0} < .... < x_{1m_n} = b_n$$

Definition 1.1 *(i)* $||P|| = \max\{|x_{ij} - x_{ij+1}| : 1 \le j \le m_i, i = 1, 2, ..., m\}$;
(ii) $M_{j_1,...,j_n} = \sup\{f(x) : x \in [x_{1\,j_1-1}, x_{1j_1}] \times ... \times [x_{nj_n-1}, x_{nj_n}]\}$;
(iii) $m_{j_1,j_2...j_n} = \inf\{f(x) : x \in [x_{1j_1-1} + x_{1j_1}] \times ... \times [x_{nj_{n-1}}, x_{nj_n}]\}$;
(iv) $L(f, P) = \sum_{j_{1=1}}^{m_1} ... \sum_{j_{n=1}}^{m_n} m_{j_1,...,j_n}(x_{ij_1} - x_{1j_1-1})...(x_{nj_n} - x_{nj_n-1})$;
(v) $U(f, P) = \sum_{j_{1=1}}^{m_1} ... \sum_{j_{n=1}}^{m_n} M_{j_1,...,j_n}(x_{ij_1} - x_{1j_1-1})...(x_{nj_n} - x_{nj_n-1})$;
(vi) $L(f) = \lim_{||P||\to 0} L(f, P)$, $U(f) = \lim_{||P||\to 0} U(f, P)$;
(vii) f is bounded on E, say f is integrable on E if $L(f) = U(f)$.

THEOREM 1.2 *(Fubini Theorem) Let f be integrable on $E = [a_1, b_1] \times ... \times [a_n, b_n]$. Then*

$$\int_E f(x)dx = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, ..., x_n)dx_n\, dx_{n-1} \cdots dx_1,$$

where dx is positive oriented volume element of $\mathbb{R}^n$.

Let D be a bounded domain on $\mathbb{R}^n$ and let f be a bounded function on D. We say that f is integrable on D if

$$F(x) = \begin{cases} f(x), & \text{if } x \in D \\ 0, & \text{if } x \notin D \end{cases}$$

is integrable on $[M, M]^n$ for $M > 0$ with $D \subset [-M, M]^n$.

EXAMPLE 1 *(Fubini them) Evaluate $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dxdy$.*

Solution Since

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} = \int_D e^{x^3} dxdy = \int_0^1 \int_0^{x^2} e^{x^3} dydx = \int_0^1 e^{x^3} x^2 dx = \frac{1}{3} e^{x^3}|_{x=0}^1 = \frac{1}{3}(e-1).$$

1.2 Formula for Change of Variables

Let $\Phi : D \to G(\subset \mathbb{R}^n)$ be 1-1, onto and differentiable with $\det \phi'(x) \neq 0$. Assume that f is integrable on G, then $f \circ \Phi|\det\Phi'(x)|$ is integrable on D. Moreover,

$$\int_D f(\Phi(x)|\det\Phi'(x)|dx = \int_G f(y)dy.$$

- In $\mathbb{R}^2$, we have the polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dA = |\det\Phi(r,\theta)|drd\theta = rdrd\theta.$$

EXAMPLE 2 *Prove* $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

Proof. Notice that

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right) \\
&= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy \\
&= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrd\theta \\
&= \int_0^{2\pi} \frac{1}{2}(-e^{-r^2})\,|_{r=0}^{\infty}\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} d\theta \\
&= \pi
\end{aligned}$$

EXAMPLE 3 *For $a, b, c > 0$, we let*

$$D(a,b,c) = \{(x,y,z) : \frac{x^2}{c^2} + \frac{y^2}{c^2} + \frac{z^2}{c^2} < 1\}.$$

Prove that the volume of $D(a,b,c) = \frac{4\pi}{3}abc$.

Proof. Let

$$(x,y,z) = \phi(\bar{x},\bar{y},\bar{z}) = (a\bar{x}, y = a\bar{y}, z = a\bar{y}).$$

Then

$$\det\phi' = abc$$

and

$$\mathrm{Vol}(D(a,b,c)) = \int_{D(a,b,c)} dV(x,y,z)$$

$$
\begin{aligned}
&= \int_{B_3(0,1)} |\det \phi'| dV(\bar{x}, \bar{y}, \bar{z}) \\
&= \int_{B_3(0,1)} abcdV \\
&= abc \operatorname{Vol}(B_3(0,1)) \\
&= \frac{4\pi}{3} abc.
\end{aligned}
$$

1.2.1 Coordinate systems in $\mathbb{R}^3$

(i) Rectangular coordinates (x, y, z);
(ii) Cylindrical coordinates (r, θ, z), $x = r\cos\theta$, $y = r\sin\theta$.
$\Phi(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$

$$
\det\ \Phi(r, \theta, z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r
$$

(iii) Spherical coordinates $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.
The Jacobian matrix of this coordinate change is:

$$
J_F(\rho, \phi, \theta) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}
$$

- The determinant of $J_F(\rho, \phi, \theta)$ is $\rho^2 \sin\phi$.

1.3 Line Integrals

Let $\Gamma = (x_1(t), ..., x_n(t)) : [a, b] \to \mathbb{R}^n$ be a piecewise C^1 curve. We define the line integral:

$$
\int_\Gamma f_1 dx_1 + \cdot + f_n dx_n = \int_a^b f_1(\Gamma(t)) x_1'(t) + \cdots + f_n(\Gamma(t)) x_n'(t) dt.
$$

Arclength of $\Gamma = \int_a^b |\Gamma'(t)| dt$.

1.4 Integrals on Surface

Let S be a smooth surface in $\mathbb{R}^n$ and let f be a continuous function on S.
Question: How to evaluate $\int_S f(x) d\sigma =$?

Case 1: If $S = \{(x_1, ..., x_{n-1}, f(x_1, ..., x_{n-1})) : (x_1, ..., x_{n-1}) \in D\}$, D is a domain in $\mathbb{R}^{n-1}$, then

$$\int_S F(x) d\sigma(x) =: \int_S F(x_1, ..., x_{n-1} f(x_1, , ..., x_{n-1})) \sqrt{1 + \sum_{j=1}^{n-1} (\frac{\partial f}{\partial x_j})^2}\, dx_1 \cdots x_{n-1}.$$

Case 2: If $S = \Phi(D)$ and $\Phi = (\phi_1, ..., \phi_n) : D \to \mathbb{R}^n$, then

$$d\sigma(t_1, ..., t_{n-1}) = \sqrt{\sum_{j=1}^{n} (\det \hat{\phi}_j(t))^2} dt_1 \cdots dt_{n-1}$$

where $\hat{\phi_j} = (\phi_1, ..., \phi_{j-1}, \phi_{j+1} ...., \phi_n)$.

EXAMPLE 4 *Find the area of the portion of the unit sphere:* $x^2 + y^2 + z^2 = 1$ *in* $\mathbb{R}^3$ *with* $z \geq \sqrt{3}/2$.

Solution Method 1 Use spherical coordinates:

$$(x, y, z) = \Phi(\theta, \phi) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi).$$

Then $z = \sqrt{3}/2$ if and only if $\cos\phi = \sqrt{3}/2$ if and only if $\phi = \pi/6$. By the formula above:

$$d\sigma(\phi, \theta) = \sqrt{(\frac{\partial(x, y)}{\partial(\phi, \theta)})^2 + (\frac{\partial(y, z)}{\partial(\phi, \theta)})^2 + (\frac{\partial(x, z)}{\partial(\phi, \theta)})^2}\, d\phi d\theta = \sin\phi\, d\phi d\theta.$$

Therefore,

Area $= \int_0^{2\pi} \int_0^{\pi/6} \sin\phi d\phi d\theta = 2\pi(-\cos\phi) \Big|_{\phi=0}^{\pi/6} = 2\pi(1 - \frac{\sqrt{3}}{2})$.

Method 2 Using rectangular coordinates (x, y, z).

Notice that

$$\{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z \geq \sqrt{3}/2\}$$

if and only if

$$z = \sqrt{1 - x^2 - y^2} : D = \{(x, y) : x^2 + y^2 \leq \frac{1}{4}\} \to \mathbb{R}^3.$$

Therefore,

$$\text{Area} \quad = \quad \int_D \int \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dx\, dy$$

$$
\begin{aligned}
&= \int_D \sqrt{1+(\frac{x^2}{1-x^2-y^2})+(\frac{y^2}{1-x^2-y^2})}dxdy \\
&= \int_D \sqrt{\frac{1}{1-x^2-y^2}}dxdy. \\
&= \int_0^{2\pi}\int_0^{1/2}\sqrt{\frac{1}{1-r^2}}rdrd\theta \\
&= 2\pi\int_0^{1/2}\sqrt{\frac{1}{1-r^2}}rdr \\
&= \pi\int_0^{1/4}\sqrt{\frac{1}{1-t}}dt \\
&= \pi\int_1^{3/4}-\frac{1}{\sqrt{t}}dt \\
&= \pi\int_{3/4}^{1}t^{-1/2}dt \\
&= 2\pi t^{1/2}|_{3/4}^{1} \\
&= 2\pi(1-\sqrt{3}/2).
\end{aligned}
$$

EXAMPLE 5 *Let $B(0,R)$ be the ball in $\mathbb{R}^n$ centered at 0 with radius R. Then*

$$\int_{B_n(0,R)} f(x)dv(dv=dx) = \int_0^R\int_{S^{n-1}} f(rx)d\sigma(x)r^{n-1}dr$$

where $S^{n-1}=\{x\in\mathbb{R}^n : ||x||=1\}$ and $d\sigma$ is surface element on S^{n-1}.

In particular,

$$Vol(B_n(0,R)) = \int_0^R\int_{S^{n-1}} d\sigma r^{n-1}dr = \sigma(S^{n-1})\int_0^R r^{n-1}dr = \frac{\sigma(S^{n-1})}{n}R^n.$$

When $n=3$, $Area(S^2)=4\pi$, $Vol(B_n(0,R))=\frac{4\pi}{3}R^3$.

1.5 Divergence theorem and Stokes theorem

For $n=1$, Newton-Leibniz's theorem gives $\int_a^b f'(x)dx = f(b)-f(a)$.

Question: What about when $n>1$?
Answer: There are Stokes theorem and Divergence theorem.

Let $F=(F_1,\cdots,F_n): D\to\mathbb{R}^n$ be a mapping. We define the divergence of F as follows:

$$\mathrm{div}(F)=\sum_{j=1}^n\frac{\partial F_j}{\partial x_j}$$

Let $D = \{x \in \mathbb{R}^n : r(x) < 0\}$ with r is a defining function of D and the $\partial D = \{x \in \mathbb{R}^n : r(x) = 0\}$ and $\nabla r(x) \neq 0$ on ∂D. Then the unit outer normal vector to ∂D at x is given by:

$$\mathbf{n}(x) = (n_1(x), \cdots, n_n(x)) = \frac{1}{\|\nabla r(x)\|}\nabla r(x).$$

Flux of F at $x \in \partial D$ is $\text{Flux}(F) = F \cdot \mathbf{n} = \sum_{j=1}^{n} F_j(x)n_j(z)$. The divergence theorem says that Total divergence of F on D is equal to the total flux of F on ∂D. Precisely, we have

THEOREM 1.3 *(Divergence Theorem) Let D be a bounded domain in $\mathbb{R}^n$ with piecewise C^1 boundary. Let $F = (F_1, \cdots, F_n) : \overline{D} \to \mathbb{R}^n$ be a differentiable map. Then*

$$\int_D div(F)dv = \int_{\partial D} \vec{F} \cdot \vec{n}(x) d\sigma.$$

EXAMPLE 6 *Let*

$$D(a,b,c) = \{(x,y,z) : r(x) =: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 < 0\}.$$

Evaluate $\int_{\partial D(a,b,c)} \frac{x^2}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} d\sigma$.

Solution. Since $X = (x,y,z) \in \partial D(a,b,c)$ and

$$\nabla r(X) = 2(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})$$

Thus,

$$\mathbf{n}(X) = \frac{\nabla r(X)}{\|\nabla r(X)\|} = \frac{1}{\sqrt{(x^2/a^4) + (y^2/b^4) + (z^2/c^4)}}(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2})$$

Let

$$F(X) = (a^2x, 0, 0)$$

Then

$$\text{div}(F) = a^2, \quad \text{Flux}(F) = \frac{x^2}{\sqrt{(x^2/a^4) + (y^2/b^4) + (z^2/c^4)}}$$

By the Divergence Theorem, we have

$$\int_{\partial D(a,b,c)} \frac{x^2}{\sqrt{(x^2/a^4) + (y^2/b^4) + (z^2/c^4)}} d\sigma(x,y,z) = \int_{D(a,b,c)} a^2 dv = \frac{4\pi}{3}a^3bc.$$

- Differential forms:

• The wedge product $\wedge$: We define

$$dx_i \wedge dx_j = -dx_j \wedge dx_i, \quad dx_i \wedge dx_i = 0$$

and

$$dx^\alpha = dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_n}$$

with $\alpha = (\alpha_1, ..., \alpha_n)$ and $\alpha_j = 0$ or 1.

• k-form: A k-form is

$$f = \sum_{|\alpha|=k} f_\alpha dx^\alpha$$

where f_α is a function; and df is $(k+1)$-form defined by

$$df = \sum_{|\alpha|=k} df_\alpha \wedge dx^\alpha = \sum_{|\alpha|=k} \sum_{j=1}^n \frac{\partial f_\alpha}{\partial x_j} dx_j \wedge dx^\alpha.$$

THEOREM 1.4 *(Stokes theorem) Let D be a bounded domain in $\mathbb{R}^n$ with piecewise C^1 boundary ∂D. Let f be differentiable $(n-1)$-form on $\overline{D}$, then $\int_{\partial D} f = \int_D df$.*

EXAMPLE 7 *Evaluate $\int_{\partial B_n} \sum_{j=1}^n (-1)^j x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n$.*

Proof. By stokes theorem, one has t

$$\begin{aligned}
&\int_{\partial B_n} \sum_{j=1}^n (-1)^j x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n \\
&= \int_{B_n} d(\sum_{j=1}^n (-1)^j x_j dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n) \\
&= \int_{B_n} \sum_{j=1}^n (-1)^j dx_j \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \cdots \wedge dx_n \\
&= \int_{B_n} \sum_{j=1}^n (-1)^j (-1)^{j-1} dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n) \\
&= \int_{B_n} (-1) \sum_{j=1}^n dx_1 \wedge \cdots \wedge dx_n \\
&= -n \int_{B_n} dV \\
&= -n \mathrm{Vol}(B_n)
\end{aligned}$$

THEOREM 1.5 *(Green's Formula) Let D be a bounded domain in $\mathbb{R}^n$ with piecewise C^1 boundary ∂D. Let f, g be twice differentiable in $\bar{D}$. Then*

$$\int_{\partial D} g D_{\mathbf{n}} f d\sigma - \int_{\partial D} f D_{\mathbf{n}} g d\sigma = \int_D g\Delta f - f\Delta g dv.$$

where
$$D_{\mathbf{n}}f(x) = \sum_{j=1}^{n} n_j(x)\frac{\partial f}{\partial x_j}.$$

Proof. Since

$$\begin{aligned}
\int_D g\Delta f - f\Delta g dv &= \int_D \operatorname{div}(g\nabla f) - \operatorname{div}(f\nabla g) dv \\
&= \int_{\partial D} \mathbf{n}\cdot g\nabla f - \mathbf{n}\cdot f\nabla g) d\sigma \\
&= \int_{\partial D} (gD_{\mathbf{n}}f - fD_{\mathbf{n}}g) d\sigma
\end{aligned}$$

This proves theorem. $\square$

1.6 Exercise

1. Let $f(x) \geq 0$ be continuous on $(0,1)$. Prove $\int_0^1 f(x)$ exists if and only if $\lim_{\epsilon\to 0+}\int_\epsilon^{1-\epsilon} f(x)dx$ exists.

2. Evaluate the following integral:
$$\int_D \Big(\sin(b^2x^2 + a^2y^2) + cx^3 + y^5\Big)dxdy$$
where
$$D = \Big\{(x,y)\in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\Big\}.$$

3. Let D be a Jordan domain in $\mathbb{R}^n$. Let $f(x)$ and $f_n(x)$ be integral on D for $n = 1, 2, 3, \cdots$. Assume that $f_n(x) \to f(x)$ as $n \to \infty$ uniformly for $x \in D$. Prove
$$\lim_{n\to\infty}\int_D f_n(x)dx = \int_D f(x)dx.$$

4. Prove
$$f(x) = \begin{cases} \|x\|^{-\alpha}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
is integrable on the ball $B(0,1)$ in $\mathbb{R}^n$ when $\alpha < n$. (If you can not do it for general n, try to prove it when $n \leq 3$.)

5. Evaluate the surface integral
$$\int_S x^2 + y^2 + (z-1)^2\, d\sigma$$

where

$$S = \Big\{(x, y, z) : x^2 + y^2 + z^2 = 1 : x > 0, y > 0, z < 0\Big\}.$$

6. Find the area of $S^3 = \Big\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\Big\}$.

7. Define $dx_i \wedge dx_j = -dx_j \wedge dx_i$ and $g(x)$ is a function of x, we define

$$dg(x) = \frac{\partial g}{\partial x_1} dx_1 + \cdots + \frac{\partial g}{\partial x_n} dx_n$$

Let $G(x_1, x_2) = (g_1(x), g_2(x))$. Prove that (here $n = 2$)

$$(dg_1) \wedge (dg_2) = \det G'(x) dx_1 \wedge dx_2.$$

8. Evaluate the integral

$$\int_{S^{n-1}} |x_1|^2 d\sigma(x)$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$.