Summer Jump-Start Program for Analysis, 2012

Song-Ying Li

1 Lecture 7: Equicontinuity and Series of functions

1.1 Equicontinuity

Definition 1.1 *Let (X, d) be a metric space, $K \subset X$ and K is a compact subset of X.*

- *$C(K)$ denotes the set of continuous functions on f on K.*
- *Let $F(K)$ be the subset of $C(K)$. Then*

(a) $F(K)$ is bounded pointwise on K if for any given $x \in K$ there is $M_x > 0$ such that $|f(x)| \le M_x$ for all $f \in F(K)$;

(b) $F(K)$ is bounded uniformly on K if there is $M > 0$ such that $|f(x)| \le M$ for all $f \in F(K)$ and $x \in K$;

(c) $F(K)$ is equicontinuous on K if for any $\epsilon > 0$, there is $\delta > 0$ such that if $d_X(x, y) < \delta$, $x, y \in K$, then $|f(x) - f(y)| < \epsilon$ for all $f \in F(K)$.

EXAMPLE 1 *(1) $\mathcal{F} = \{\frac{1}{nx} : n = 1, 2, ...\}$ is pointwise bounded on $K = (0, 1]$, but not bounded on K;*

(2) $\mathcal{F} = \{nx : n = 1, 2, ...\}$ is not equicontinuous on $[0, 1]$;

(3) $\mathcal{F} = \{f_\alpha : \alpha \in \bigwedge\}$ with $|f'_\alpha(x)| \le 1$ $x \in K = [a, b]$, $\alpha \in \Lambda$ then F is equicontinuous on $[a, b]$.

Proof. (1) and (2) are straight forward. Now we prove (3). For any $\epsilon > 0$, let $\delta = \epsilon$. When $x, y \in [a, b]$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |f'(\xi)||x - y| \le 1 \cdot |x - y| \le \delta = \epsilon$$

By deifnition, F is equicontinuous on $[a, b]$. □

THEOREM 1.2 *(Arzelà - Ascoli Theorem) Let F be a family of continuous functions on $[a, b] \subset \mathbb{R}^1$. Then the following two statements are equivalent:*

(a) F is equi-continuous on $[a, b]$ and F is bounded pointwise on $[a, b]$.

(b1) F is bounded on $[a, b]$ and

(b2) every sequence $\{f_n\}_{n=1}^\infty \in F$ has a uniformly convergent subsequence $\{f_{n_k}\}$ on $[a, b]$.

Proof. b) ⇒ a)

(i) By (b1), F is bounded pointwise on $[a, b]$.

Suppose that F is not equicontinuous on $[a, b]$, then there exists $\epsilon_0 > 0$ such that for any $\delta = \frac{1}{k}(k \in \mathbb{N})$, there are two points $x_k, y_k \in [a, b]$ with $|x_k - y_k| < \delta = \frac{1}{k}$, and there is $f_k \in F$ such that $|f_k(x_k) - f_k(y_k)| \ge \epsilon_0$.

By (b2), $\{f_k\}_{n=1}^{\infty}$ has a subsequence $\{f_{k_\ell}\}$ and f on $[a,b]$ such that $f_{k_\ell} \to f$ uniformly on $[a,b]$ as $\ell \to \infty$. Therefore, $f \in C([a,b])$. We may choose a convergent subsequence $\{x_{k_{\ell_j}}\}$ of $\{x_{k_\ell}\}$, without loss of generality, we may assume $x_{k_\ell} \to x \in [a,b]$ for some x. Since $|x_{k_\ell} - y_{k_\ell}| < \frac{1}{k_\ell} \to 0$ as $\ell \to \infty$. So, $y_{k_\ell} \to x$ as $\ell \to \infty$. Therefore, when $\ell \to \infty$, one has

$$\epsilon_0 \le |f_{k_\ell}(x_{k_\ell}) - f_{k_\ell}(y_{k_\ell})| \le |f_{k_\ell}(x_{k_\ell}) - f(x_{k_\ell})| + |f(x_{k_\ell}) - f(y_{k_\ell})| + |f(y_{k_\ell}) - f_{k_\ell}(y_{k_\ell})| \to 0.$$

This is a contradiction. So, F is equicontinuous.

Next, we prove a) $\Rightarrow$ b.)

(i) It is easily to prove that (a) implies (b1).

(ii) Next we prove (a) implies (b2). For any sequence $\{f_n\}_{n=1}^{\infty} \subset F$, we need to choose a subsequence $\{f_{n_k}\}$ and f such that $f_{n_k} \to f$ uniformly on $[a,b]$.

Step 1: Choose a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges pointwise on $\mathbb{Q} \cap [a,b]$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{Q}$ is countable. Thus, $\mathbb{Q} \cap [a,b] = \{x_1, x_2, ..., x_{n_1}, ...\}$.

Since $\{f_n(x_1)\}_{n=1}^{\infty}$ is bounded sequence in $\mathbb{R}$. There is a convergent subsequence: $\{f_{1,n}(x_1)\}_{n=1}^{\infty}$ with $f_{1,n}(x_1) \to f(x_1)$ (some number in $\mathbb{R}$, we call it $f(x_1)$).

Chose a subsequence $\{f_{2,n}(x)\}$ from $\{f_{1,n}(x)\}$ such that $f_{2,n}(x_2) \to f(x_2) \in \mathbb{R}$,. In particular, $f_{2,n}(x_1) \to f(x_1)$.

Continuing in this fashion, choose $\{f_{k,n}(x)\}$ from $\{f_{(k-1),n}\}$ such that $f_{k,n}(x_j) \to f(x_j)$ $j \le k$ as $n \to \infty$ for all $k = 2, 3, \cdots$.

Let $f_{n_k}(x) = f_{k,k}(x)$. Then $\{f_{n_k}\}$ is a subsequence of $\{f_n\}$ with $f_{n_k}(x_j) \to f(x)$ as $k \to \infty$ for each $x \in \mathbb{Q} \cap [a,b]$.

Step 2: Since $\{f_{n_k}\}$ is a family of equicontinuous on $[a,b]$, one limit function $f(x)$ is uniformly continuous on $\mathbb{Q} \cap [a,b]$.

For any $\epsilon > 0$, since $\{f_{n_k}\}$ is equicontinuous on $[a,b]$, there is $\delta > 0$ such that for any $x, y \in [a,b]$ with $|x-y| < \delta$, one has

$$|f_{n_k}(x) - f_{n_k}(y)| < \epsilon, \quad k \in \mathbb{N}.$$

For any $x, y \in [a,b] \cap \mathbb{Q}$ with $|x-y| < \delta$, there is $k >> 1$ such that $|f_{n_k}(x) - f(x)|, \quad |f_{n_k}(y) - f(y)| < \epsilon$. Thus,

$$|f(x) - f(y)| \le |f_{n_k}(x) - f(x)| + |f_{n_k}(x) - f_{n_k}(y)| + |f_{n_k}(y) - f(y)| \le 3\epsilon.$$

Therefore, f is uniformly continuous on $[a,b] \cap \mathbb{Q}$. So, we can extend f as a continuous function on $[a,b]$. For the $\epsilon > 0$, there is $\delta > 0$ such that if $|x-y| < \delta$, we have

$$|f(x) - f(y)| < \epsilon.$$

Choose rational numbers $\{r_1, \cdots, r_m\} \subset [a,b]$ such that for any $x \in [a,b]$, there is $r_k \in \{r_1, \cdots, r_m\}$ such that $|x - r_k| < \delta$. Thus for any $x \in [a,b]$, we choose

$r_j \in \{r_1, \cdots, r_m\}$ such that $|r_j - x| < \delta$. There is k_0 such that $k \geq k_0$, we have $|f_{n_k}(r_j) - f(r_j)| < \epsilon$ for all $j = 1, 2, \cdots, m$. Therefore,

$$|f_{n_k}(x) - f(x)| \leq |f_{n_k}(x) - f_{n_k}(x_j)| + f(r_j) - f_{n_k}(x_j)| + f(x_j) - f_{n_k}(x)| < 3\epsilon,$$

So, $f_{n_k} \to f$ uniformly on $[a, b]$ as $k \to \infty$. ∎

1.2 Series of functions

Let (X, d) be a metric space or $\mathbb{R}^n$ and let $K \subset X$. We consider a sequence of functions $\{f_n(x)\}_{x=1}^{\infty}$ on K. Let

$$s_n(x) = \sum_{k=1}^{n} f_k(x), \quad x \in K, n = 1, 2, 3, \cdots.$$

Definition 1.3 *(a) $\sum_{n=1}^{\infty} f_n(x)$ converges on K if $\lim_{n\to\infty} s_n(x)$ exists for each $x \in K$.*

(b) $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely on K if $\sum_{n=1}^{\infty} |f_n(x)|$ converges on K.

(c) $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on K if $\{s_n\}_{n=1}^{\infty}$ uniform Cauchy sequence on K.

- **Question:** How to test if a series converges uniformly on a set K?

THEOREM 1.4 *(Weirstrass M-test) Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of functions on K such that*

$$|f_n(x)| \leq M_n, \quad x \in K, \ n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} M_n < +\infty$, then $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely and uniformly on K.

Proof. For the $\epsilon > 0$, we need to fin N such that if $m > n \geq N$, then $\sum_{k=n}^{m} |f_k(x)| < \epsilon$, $x \in K$.

Since $\sum_{k=1}^{\infty} M_k < +\infty$, for $\epsilon > 0$, there is N s.t. if $m > n \geq N$, then $\sum_{k=n}^{m} M_k < \epsilon$.

Therefore, when $m > n \geq N$, $\sum_{k=n}^{m} |f_k(x)| \leq \sum_{k=n}^{m} M_k < \epsilon$, $x \in K$. Thus, $\sum_{k=1}^{\infty} |f_k(x)|$ converges absolutely and uniformly on K.

EXAMPLE 2 *Determine if $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly for $x \in \mathbb{R} = (-\infty, \infty)$.*

Solution. We claim: $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly on $\mathbb{R}$. Since

$$\left| \frac{\sin(nx)}{n^2} \right| \leq \frac{1}{n^2}, \quad x \in \mathbb{R}, \quad \text{for all } n = 1, 2, ...$$

Notice that $\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$ (p-series with $p = 2$). By Weierstrass M-test, we have $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges absolutely and uniformly on $\mathbb{R}$.

EXAMPLE 3 *Find* $f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$, $x \in (-\infty, \infty)$.

Solution Since

$$f(0) = \sum_{n=1}^{\infty} \frac{0^2}{(1+0)^n} = 0.$$

For any $x \neq 0$, since

$$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} \left(\frac{1}{1+x^2}\right)^n = x^2 \frac{\frac{1}{1+x^2}}{1-\frac{1}{1+x^2}} = \frac{x^2}{1+x^2}\frac{1+x^2}{x^2} = 1.$$

Therefore

$$f(x) = \begin{cases} 0 & x = 0, \\ 1 & x \neq 0. \end{cases}$$

EXAMPLE 4 *Discuss* $f(x) =: \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$ *by answering the following questions.*

(a) Where does the series converge? i.e., where is $f(x)$ well-defined?
(b) Where does the series converge uniformly?
(c) Where does the series not converge uniformly?
(d) Where is $f(x)$ continuous?

Solution Since $f_n(x)$ is well-defined on $\mathbb{R}\backslash\{-1/n^2\}$ and it is continuous there. Let

$$K = (-\infty, \infty)\backslash\{-1/n^2 : n = 1, 2, ...\}.$$

(a) when $x = 0$, $f_n(0) = 1 \not\to 0$, so series does not converge at $x = 0$, and $f(0) = +\infty$. Let $K_0 = K\backslash\{0\}$. We claim $\sum_{n=1}^{\infty} f_n(x)$ converges on K_0.

For any $x_0 \in K_0$, we divide it into two cases:

(i) $x_0 > 0$, $f_n(x_0) = \frac{1}{1+n^2 x_0} \leq \frac{1}{x_0} \cdot \frac{1}{n^2}$. By the comparison test, $\sum_{n=1}^{\infty} f_n(x_0)$ converges.

(ii) $x_0 < 0$. There is N such that $1 + n^2 x_0 < 0$ if $n \geq N$. Thus,

$$|f_n(x_0)| = \left|\frac{1}{1+n^2 x_0}\right| \leq \frac{1}{n^2|x_0| - 1} \leq \frac{2}{n^2|x_0|} = \frac{2}{|x_0|} \cdot \frac{1}{n^2}, \quad n \geq N.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ by comparison test, $\sum_{n=1}^{\infty} f_n(x_0)$ converges absolutely.

(b) $\sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$ converges uniformly on $\{x \in \mathbb{R} : |x| \geq \delta\} \cap K_0$ for any fixed $\delta > 0$.

Proof. For any $\delta > 0$, there is N such that if $n \geq N$ then $n^2\delta \geq \frac{n^2\delta}{2} + 1$. Thus, for any $n \geq N$, $|x| \geq \delta$, $x \in K_0$, we have

(i) if $x \geq \delta$, then $|f_n(x)| = \left|\frac{1}{1+n^2 x}\right| \leq \frac{1}{n^2 x} \leq \frac{1}{n^2 \delta}$;

(ii) If $x < -\delta$ and $n > 1/\delta$, then

$$|f_n(x)| = \left|\frac{1}{1+n^2 x}\right| \leq \frac{1}{n^2|x| - 1} \leq \frac{1}{\frac{n^2\delta}{2} + 1 - 1} \leq \frac{2}{n^2\delta}.$$

So $\sum_{n=1}^{\infty} |f_n(x)|$ converges uniformly on $\{|x| \geq \delta\}$. The proves the claim (b). ▯

(c) For any $\delta > 0$, $\sum_{n=1}^{\infty} f_n(x)$ does not converge uniformly on $(-\delta, \delta) \cap K_0$.

Proof. Let $\epsilon_0 = 1/2$. For $N > 1$, $m = n + 1$, $n = N$; $x_N = \frac{1}{(1+N)^2} \in (-\delta, \delta)$.

$$\left| \sum_{k=n+1}^{m} f_k(x_N) \right| = |f_{N+1}(x_N)| = \frac{1}{1 + (N+1)^2 \frac{1}{(N+1)^2}} = \frac{1}{2} = \epsilon_0$$

So, $\sum_{n=1}^{\infty} f_n(x)$ does not converge uniformly on $(-\delta, \delta) \cap K_0$.

(d) We know by part (a) that $f(x)$ is well-defined on K_0. We know $f_n(x)$ is continuous on K_0 for $n = 1, 2, 3...$, we also know by (b) that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $K_0 \cap \{x \in \mathbb{R} : |x| \geq \delta\}$ for any given $\delta > 0$. This implies that $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is continuous on $K_0 \cap \{|x| \geq \delta\}, \delta > 0$. Therefore, $f(x)$ is continuous on $K_0 \cap \cup_{\delta > 0} \{|x| \geq \delta\} = K_0 \cap \mathbb{R} \setminus \{0\} = K_0$.

EXAMPLE 5 *Let* $f(x) =: \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$, *where* (x) *is fractional part of* x. *i.e.* $(x) = x - [x]$, $[x] =$ *integer part of* x. *Discuss where* $f(x)$ *is continuous.*

Solution Since

(i) $f_n(x) = \frac{(nx)}{n^2} = \frac{nx - [nx]}{n^2}$ is continuous on $K =: \mathbb{R} \backslash \{x = \frac{k}{n}, k \in \mathbb{Z}\} = \mathbb{R} \setminus \mathbb{Q}$ for all $n \geq 1$.

(ii) $\frac{|(nx)|}{n^2} \leq \frac{1}{n^2}$, for all $n = 1, 2, ...$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. By the Weierstrass M-Test, we know that $f(x)$ is well-defined on $\mathbb{R}$, and $f(x)$ is continuous on K.

THEOREM 1.5 *(Stone-Weierstrass Theorem) Let* $f(x)$ *be continuous on* $[a, b]$. *Then there is a sequence of polynomials* $p_n(x)$ *such that* $p_n(x) \to f(x)$ *uniformly on* $[a, b]$.

1.3 Exercise

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

2. If sequences of functions $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E.
   (a) Prove that $\{f_n + g_n\}$ converges uniformly on E.
   (b) Does $\{f_n g_n\}$ converge uniformly on E?

3. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x^2}$$

For what values of x does the series converges absolutely? On what intervals does it converges uniformly ? On what interval does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

4. Let

$$f(x) = \begin{cases} 0, & \text{if } x < \frac{1}{n+1}, \\ \sin^2(\frac{\pi}{x}), & \frac{1}{n+1} \le x \le \frac{1}{n} \\ 0, & \text{if } \frac{1}{n} < x \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum_{n=1}^{\infty} f_n$ to show that absolute convergence, even for all x, does not implies uniform convergence.

5. Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

6. If $I(x) = 0$ if $x \le 0$ and $I(x) = 1$ if $x > 0$. Let $\{x_n\}$ be a sequence of distinct points of (a, b), and if $\sum_{n=1}^{\infty} |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \le x \le b)$$

converges uniformly on $[a, b]$, and that f is continuous for every $x \ne x_n$.

7. Letting (x) denote the fractional part of the real number x, consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}, \quad x \in \mathbb{R}$$

Find all discontinuities of f, and show that they form a countable dense set.

8. Suppose $\{f\}, \{g_n\}$ are two sequences of functions defined on set E, and
   (a) $\sum_{n=1}^{\infty} f_n$ has uniformly bounded partial sums;
   (b) $g_n \to 0$ uniformly on E;
   (c) $g_1(x) \ge g_2(x) \ge g_3(x) \ge \cdots$ for every $x \in E$.
   Prove that $\sum_{n=1}^{\infty} f_n g_n$ converges uniformly on E.

9. Prove or disprove $\{f_n(x) = x^n : n \in \mathbb{N}\}$ is equicontinuous on $[0, 1)$.