Summer Jump-Start Program for Analysis, 2013

Song-Ying Li

1 Lecture 8: **Differentiation on functions of one variable**

1.1 Definition and basic property

We will discuss differentiability for functions on $\mathbb{R}$, then on $\mathbb{R}^n$.

Definition 1.1 *Let $f(x)$ be a function defined on $[a,b]$. For $x_0 \in (a,b)$, we say that $f(x)$ is differentiable at x_0 if $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists and is called $f'(x_0)$. In other word, f is differentiable at x_0 if there is a real number $f'(x_0)$ such that*

$$\lim_{x\to x_0} \frac{|f(x)-f(x_0)-f'(x_0)(x-x_0)|}{|x-x_0|} = 0.$$

From the definition, one can easily see:

Proposition 1.2 *If $f(x)$ is differentiable at x_0, then $f(x)$ is continuous at x_0.*

- Basic properties for computing derivatives:

Proposition 1.3 *Let f and g be differentiable at x. Then*
(i) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$;
(ii) Product rule: $(fg)' = f'(x)g(x) + f(x)g'(x)$;
(iii) Quotient rule: $(f/g)' = \frac{f'g-g'f}{g^2}$;
(iv) Chain rule: $(f \circ g)' = (f(g))' = f'(g)g'(x)$.

EXAMPLE 1 *Let $f(x) = x\sin(1/x)$ if $x \neq 0$ and 0 if $x = 0$. Discuss the differentiability of $f(x)$ on $\mathbb{R}$.*

Solution: (i) $f(x)$ is not differentiable at $x = 0$.
Since

$$\lim_{x\to 0} \frac{f(x)-f(x)}{x-0} = \lim_{x\to 0} \frac{x\sin(1/x)-0}{x-0} = \lim_{x\to 0} \sin(1/x) \text{ DNE}.$$

So f is not differentiable at $x = 0$.
(ii) $f(x)$ is differentiable on $\mathbb{R} \setminus \{0\}$ since x and $\sin(1/x)$ are differentiable and product rule.

EXAMPLE 2 *$g(x) = x^2\sin(1/x)$ if $x \neq 0$ and 0 if $x = 0$.*

Note that $g'(x) = 2x\sin(1/x) - \cos(1/x)$ if $x \neq 0$, 0 if $x = 0$, we have that $g'(x)$ exists at every point $x \in \mathbb{R}$, but $g'(x)$ is not continuous at $x = 0$.

1.2 Mean value theorems

THEOREM 1.4 *(Rolle's Theorem) If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there is $x_0 \in (a, b)$ s.t. $f'(x_0) = 0$.*

Proof. Since f is continuous on $[a, b]$, we have that $f(x)$ has maximum and minimum over $[a, b]$. Since $f(a) = f(b)$, there is $x_0 \in (a, b)$ such that $f(x_0)$ is either maximum or minimum of f on $[a, b]$. WLOG, we may assume that $f(x_0)$ is maximum. Then

$$\begin{aligned} f'(x_0) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \to x_0, x > x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \\ &= \lim_{x \to x_0, x < x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0. \end{aligned}$$

Therefore, $f'(x_0) = 0$. □

THEOREM 1.5 *(Lagrange Mean Value Theorem) Let $f(x)$ be continuous on $[a, b]$, $f'(x)$ exists for all $x \in (a, b)$. Then, there is a $x_0 \in (a, b)$ such that*

$$f(b) - f(a) = f'(x_0)(b - a) \quad \left(\text{or } f'(x_0) = \frac{f(b) - f(a)}{b - a} \right)$$

Proof. Let

$$g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right)$$

Note that $g \in C[a, b]$, g is differentiable on (a, b), and $g(a) = g(b) = 0$. By Rolle's Theorem., there is a $x_0 \in (a, b)$ such that$g'(x_0) = 0$. Thus, $g'(x_0) = f'(x_0) - \frac{f(b) - f(a)}{b - a} = 0$.

The more general version of the mean value theorem is as follows:

THEOREM 1.6 *Let f and g be continuous on $[a, b]$ and f and g are differentiable on (a, b). Then there is $x_0 \in (a, b)$ such that $(g(b) - g(a))f'(x_0) = (f(b) - f(a))g'(x_0)$.*

Hint: Let $H(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. Then $H(b) - H(a) = 0$. The theorem follows from Rolle's theorem.

Proposition 1.7 *Let $f(x)$ be differentiable on (a, b). Then*
(a) If $f'(x) \geq 0$ on (a, b), then $f(x)$ is increasing on (a, b);
(b) If $f'(x) \leq 0$ on (a, b), then $f(x)$ is decreasing on (a, b);
(c) If $f'(x) = 0$ on (a, b), then $f(x)$ is constant on (a, b).

Note: We learned before: if f is continuous on $[a,b]$, then for any λ between $f(a)$ and $f(b)$, there is $c \in (a,b)$ s.t. $f(c) = \lambda$. This is the intermediate value theorem. For $f'(x)$, without assuming $f'(x)$ is continuous, we still have an intermediate value for $f'(x)$.

THEOREM 1.8 *Let $f'(x)$ be differentiable on $[a,b]$. If $f'(a) < f'(b)$, then for any λ with $f'(a) < \lambda < f'(b)$, there is $c \in (a,b)$ such that $f'(c) = \lambda$.*

Proof. Let

$$g(x) = f(x) - \lambda(x-a).$$

Then $g'(a) = f'(a) - \lambda < 0$, there is $x_1 > a$ such that $g(x_1) < g(a)$. Notice that $g'(b) = f'(b) - \lambda > 0$, there is $x_2 < b$ such that $g(x_2) < g(b)$. These imply there is $x_0 \in (a,b)$ such that $g(x_0)$ is the minimum of g on $[a,b]$. Therefore, $g'(x_0) = 0$ or $f'(x_0) = \lambda$. $\square$

• (Banach fixed point theorem) Let X be a complete metric space. Let $f : X \to X$ be a contractive map: there is $c \in (0,1)$ s.t. $d(f(x), f(y)) < cd(x,y)$, $x, y \in X$. Then, $f(x)$ must be a fixed point s.t. $\exists x_0 \in X$ such that $f(x_0) = x_0$.
Remark: $c < 1$ is necessary in Banach fixed point theorem or the theorem fails when $c = 1$.

EXAMPLE 3 *(Counterexample) Let $f(x) = x + \frac{1}{1+e^x}$ and $X = \mathbb{R}$. Then*

$$f'(x) = 1 - \frac{e^x}{(1+e^x)^2} \in (0,1).$$

and thus

$$|f(x) - f(y)| = |f'(\xi)||x-y| = \left|1 - \frac{e^\xi}{(1+e^\xi)^2}\right||x-y| < |x-y|.$$

But $f(x) = x$ has no solution in $\mathbb{R}$.

THEOREM 1.9 *(L'Hospital's Rule) Let $f(x)$ and $g(x)$ be differentiable in (a,b) and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. If $g'(x) \neq 0$ and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = A$.*

EXAMPLE 4 *Find* $\lim_{x\to 0} \frac{\sin(x^2)}{x^2}$.

Solution $\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{x\to 0} \frac{\cos(x^2)2x}{2x} = \lim_{x\to 0} \frac{\cos(x^2)}{1} = 1.$

EXAMPLE 5 *Find* $\lim_{x\to 0} x \ln \frac{1}{x}$.

Solution

$$\lim_{x\to 0} x\ln(1/x) = \lim_{x\to 0} \frac{x}{1/\ln(x)} = \lim_{x\to 0^+} \frac{\ln x}{(1/x)} = \lim_{x\to 0^+} \frac{(1/x)}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$$

THEOREM 1.10 *(Taylor Theorem) Let $f(x)$ be differentiable up to order $(n+1)$ on $[a,b]$. Then there is $\xi \in (a,b)$ such that*

$$f(b) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(b-a)^j + \frac{f^{n+1}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

Proof. Let M be the number such that $f(b) - \sum_{j=0}^{n+1} \frac{f^{(j)}(a)}{j!}(b-a)^j = M(b-a)^{n+1}$. Then, we define

$$g(x) = f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j - M(x-a)^{n+1}.$$

We have $g(a) = 0$, $g(b) = 0$. By Rolle's Theorem, there is $x_1 \in (a,b)$ such that $g'(x_1) = 0$. Compute:

$$\begin{aligned} g'(x) \quad &= f'(x) - \textstyle\sum_{j=0}^{n} \frac{f^{(j)}(a)}{(j-1)!}(x-a)^{j-1} - (n+1)M(x-a)^n \\ &= f'(x) - \textstyle\sum_{j=0}^{n-1} \frac{f^{(j+1)}(a)}{j!}(x-a)^{j} - (n+1)M(x-a)^n \end{aligned}$$

Then we get $g'(a) = 0$ and again by Rolle's Thm, there is $x_2 \in (a, x_1)$ s.t. $g''(x_2) = 0$.

$$g''(x) = f''(x) - \sum_{j=0}^{n-2} \frac{f^{(j+2)}(a)}{j!}(x-a)^j - n(n+1)M(x-a)^{n-1}$$

Then we have $g''(a) = 0$, and again by Rolle's Thm $\exists\, x_3 \in (a, x_2)$ s.t. $g'''(x_3) = 0$. Continuing in this fashion, we see that $g^{(n)}(a) = 0$ and there is $x_n \in (a, x_{n-1})$ s.t. $g^{(n)}(x_n) = 0$. By Rolle's thm again, there is $\xi \in (a, x_n)$ s.t. $g^{(n+1)}(\xi) = 0$. Since $g^{(n+1)}(x) = f^{(n+1)}(x) - (n+1)! \cdot M$, one has $f^{(n+1)}(\xi) - (n+1)!M = 0$ and $M = \frac{f^{(n+1)}(\xi)}{(n+1)!}$. By the definition of M, one has $f(b) = \sum_{j=0}^{n} \frac{f^(j)(a)}{j!}(b-a)^j + \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}$. ∎

Definition 1.11 *We say that $f(x)$ is analytic at x_0 if $f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$ for $|x - x_0| < \delta$ for some $\delta > 0$. If f is analytic at every point in $[a,b]$, we say that f is real analytic in $[a,b]$, denote it as: $f \in C^\omega[a,b]$.*

Definition 1.12 *$f \in C^k(a,b)$ if $f, f', ..., f^{(k)} \in C(a,b)$*

Definition 1.13 *$f \in C^\infty(a,b)$ if $f \in C^k(a,b)$ for all $k = 0, 1, 2....$*

EXAMPLE 6 *Find a functionf $\in C^\infty(-\infty, \infty)$ near $x = 0$, but f is not real analytic at $x = 0$.*

Solution Let

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since $1/x^2 \in C^\infty((0,\infty) \cup (-\infty, 0))$, we have $f(x) = e^{-1/x^2} \in C^\infty((0,\infty) \cup (-\infty, 0))$. In order to prove $f \in C^\infty(-\infty, \infty)$. It suffices to prove $f^{(1)}(0) = 0$, $j = 0, 1, 2, ...$ This follows from the fact that

$$\lim_{x \to 0} \frac{1}{x^{2m}} \exp(-1/x^2) = 0.$$

But, $f(x)$ is not analytic around $x = 0$. Suppose yes, we try to have a contradiction. If $f(x) = \sum_{j=0}^\infty \frac{f^{(j)}(0)}{j!} x$ for $|x| < \delta$ for some $\delta > 0$. Since $f^{(j)}(0) = 0$ for $j = 0, 1, 2...$, we have $f(x) = 0$, $|x| < \delta$, which is a contradiction since $f(x) = e^{-1/x^2} \neq 0$ when $x \neq 0$.

EXAMPLE 7 *Suppose $f(x)$ is defined in $(x-\delta, x+\delta)$ and $f''(x)$ exists. Show*

$$\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0).$$

Proof. By l'Hospital's rule, one has

$$\begin{aligned}
&\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} \\
= \ & \lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0 - h)}{2h} \\
= \ & \lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0) + f(x_0) - f'(x_0 - h)}{2h} \\
= \ & \frac{1}{2} \lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0)}{h} + \frac{1}{2} \lim_{h \to 0} \frac{(f'(x_0 - h) - f'(x_0)}{-h} \\
= \ & \frac{1}{2} f''(x_0) + \frac{1}{2} f''(x_0) \\
= \ & f''(x_0).
\end{aligned}$$

□

• More on convex functions

THEOREM 1.14 *(a) If $f''(x)$ exists, $x \in [a, b]$, then $f(x)$ is convex on $[a, b]$ $\Leftrightarrow f''(x) \geq 0$;*

(b) If $f'(x)$exist on (a, b), then f is convex on (a, b) if and only if

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0), \quad x_0, \ x \in (a, b).$$

Proof. "$\Leftarrow$" Assume $f''(x) \geq 0$ on (a,b). Show that $f(x)$ is convex on (a,b).

For any x_1, x_2, $(x_1 < x_2) \in [a,b]$, $\lambda \in [0,1]$, there are $(\xi_1 \in (x_1, \lambda x_1 + (1-\lambda)x_2)$ and $\xi_2 \in (\lambda x_1 + (1-\lambda)x_2, x_2)$ and $\xi_3 \in (\xi_1, \xi_2)$ such that

$$
\begin{aligned}
&\lambda f(x_1) + (1-\lambda)f(x_2) - f(\lambda x_1 + (1-\lambda)x_2) \\
&\quad = \lambda(f(x_1) - f(\lambda x_1 + (1-\lambda)x_2)) + (1-\lambda)(f(x_2) - f(\lambda x_1 + (1-\lambda)x_2)) \\
&\quad = \lambda f'(\xi)((1-\lambda)x_1 - (1-\lambda)x_2) + (1-\lambda)f'(\xi_2)(-\lambda x_1 + \lambda x_2) \\
&\quad = \lambda(1-\lambda)f'(\xi_1)(x_1 - x_2) + (1-\lambda)\lambda f'(\xi_2)(x_2 - x_1) \\
&\quad = \lambda(1-\lambda)(x_2 - x_1)[f'(\xi_2) - f'(\xi_1)] \\
&\quad = \lambda(1-\lambda)(x_2 - x_1)f''(\xi_3)(\xi_2 - \xi_1) \\
&\quad \geq 0
\end{aligned}
$$

Therefore, $f(\lambda x_1 + (1-\lambda x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$. So $f(x))$ is convex on (a,b).

Conversely, for any $x_0 \in (a,b)$, since

$$f(x_0 + h) + f(x_0 - h) - 2f(x_0) \geq 0,$$

we have

$$f''(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} \geq 0$$

$\square$

EXAMPLE 8 *Let $f(x)$ be continuous on $(0,\infty)$, and let $f'(x)$ exist on $(0,\infty)$ with $f'(x)$ is monotone increasing and $f(0) = 0$. Show $g(x) = \frac{f(x)}{x}$ is increasing on $(0,\infty)$.*

Proof. Let $x_1, x_2 \in (0,\infty)$ with $x_1 < x_2$. We consider

$$
\begin{aligned}
g(x_2) - g(x_1) &= \frac{f(x_2)}{x_2} - \frac{f(x_1)}{x_1} \\
&= \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 x_2} \\
&= \frac{x_1(f(x_2) - f(x_1)) + f(x_1)(x_1 - x_2)}{x_1 x_2} \\
&= \frac{1}{x_1 x_2}[x_1(f(x_2) - f(x_1) + (f(x_1) - f(0)(x_1 - x_2))] \\
&= \frac{1}{x_1 x_2}[x_1 f'(\xi_2)(x_2 - x_1) + f'(\xi_1)x_1(x_1 - x_2)] \\
&= (f'(\xi_2) - f'(\xi_1))\frac{x_1(x_2 - x_1)}{x_1 x_2} \\
&\geq 0
\end{aligned}
$$

Here: $\xi_1 \in (0, x_1), \xi_2 \in (x_1, x_2)$, $\xi_1 < \xi_2$ and f' is monotone increasing.

EXAMPLE 9 *Let f, f', f'' be bounded on $[0,\infty)$. Let $M_j = \sup\{|f^{(j)}(x)| : x \in [-\infty,\infty)\}$, then $M_1^2 \leq 4M_0M_2$ and the inequality is sharp.*

Proof. For $x \in (0,\infty)$ and $h > 0$, by Taylor Theorem, there is $\xi \in (x, x+2h)$ such that

$$f(x+2h) = f(x) + f'(x)2h + \frac{1}{2!}f''(\xi)(2h)^2$$

This implies

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - f''(\xi)h.$$

Thus,

$$|f'(x)| \leq \frac{1}{2h}[2M_0] + M_2h.$$

Therefore, $M_1 \leq \frac{1}{h}M_0 + hM_2$ for all $h > 0$. Let $g(h) = \frac{M_0}{h} + M_2h$ on $(0,\infty)$. If $g'(h) = -\frac{M_0}{h^2} + M_2 = 0$ then $h^2 = M_0/M_2$.

Therefore, g attains its minimum at $h = \sqrt{M_0/M_2}$. Thus, $M_1 \leq \sqrt{M_0M_2} + M_2\sqrt{M_0/M_2} = 2\sqrt{M_0}\sqrt{M_2}$. Thus, $M_1^2 \leq 4M_0M_2$.

EXAMPLE 10 *Let $f(x) = 2x^2 - 1, -1 < x < 0$; $f(x) = \frac{x^2-1}{x^2+1}, 0 \leq x < \infty$. Then $M_1^2 = 4M_0M_2$.*

Proof. It is easy to see that $M_0 = 1$.

$$f'(x) = 4x, \text{ for } x \in (-1,0); \quad f'(x) = \frac{4x}{(x^2+1)^2} \text{ when } x \geq 0;$$

and

$$f''(x) = 4, \text{ for } x \in (-1,0); \quad f''(x) = \frac{4(1-3x^2)}{(x^2+1)^3} \text{ when } x \geq 0;$$

Therefore, $M_1 = 4$ and $M_2 = 4$. Therefore, $M_1^2 = M_0M_2$. ∎

1.3 Exercise

1. If $f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove $f(x)$ is constant

2. If $c_0 + \frac{c_1}{2} + \cdots + \frac{c_n}{n+1} = 0$ with $c_0, c_1, \cdots, c_n$ are real numbers. Prove $f(x) = \sum_{j=0}^{n} c_jx^j$ has at least one zero in $(0,1)$.

3. Assume that $f(x)$ is continuous on $[0,\infty)$ such that $f(0) = 0$ and $f'(x)$ exists and monotone increasing on $(0,\infty)$. Prove $g(x) = \frac{f(x)}{x}$ is monotonically increasing on $(0,\infty)$.

4. Let $f'(x)$ be continuous on $[a,b]$. Prove that for any $\epsilon > 0$, there is $\delta > 0$ so that if $x, y \in [a,b]$ satisfy $|x-y| < \delta$ then

$$\Big| \frac{f(y) - f(x)}{y - x} - f'(x) \Big| < \epsilon.$$

5. Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

6. Let f be twice-differentiable on $(0,\infty)$ so that $f''(x)$ is bounded on $(0,\infty)$. Prove that if $\lim_{x\to\infty} f(x) = 0$ then $\lim_{x\to\infty} f'(x) = 0$.

7. Let f be real-valued, three times differentiable on $[-1,1]$ so that

$$f(-1) = f(0) = f'(0) = 0 \quad \text{and} \quad f(1) = 1.$$

Prove that there is $x_0 \in (-1,1)$ so that $f^{(3)}(x_0) \geq 3$.

8. Find $\lim_{x\to 0} \frac{\sin^2(x^3)}{x^6+x^7}$.

8. Let E be a closed subset in $\mathbb{R}$. Prove or disprove there is a $f \in C^\infty(\mathbb{R})$ so that the zero set of $Z(f) = \{x \in \mathbb{R} : f(x) = 0\} = E$.

9. Let $f(x)$ be real analytic function on $\mathbb{R}$ so that there is a bounded infinite set F so that $f(x) = 0$ if $x \in F$. Prove $f(x) \equiv 0$ on $\mathbb{R}$.

10 Construct a real analytic function f on $\mathbb{R}$ so that the radius of the convergence for the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(k)}{n!}(x-k)^n$$

is one for all non-negative integers k.