Summer Jump-Start Program for Analysis, 2013

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1 Lecture 9: Inverse and Implicite Function Theorems in $\mathbb{R}^n$

1.1 Differentiation of functions in $\mathbb{R}^n$

Definition 1.1 *Let $f(x)$ be a function on $\mathbb{R}^n$, and let $x_0 \in \mathbb{R}^n$. Then the partial derivatives of f at x_0 with respect to x_j, $j = 1, 2, \cdots, n$ are defined as follows:*

$$\frac{\partial f}{\partial x_j}(x_0) = \lim_{h\to 0} \frac{f(x_0 + he_j) - f(x_0)}{h},$$

where $e_j = (0, ..., 1, ...0)$, $j = 1, 2, \cdots, n$.

EXAMPLE 1 *Let $f(x, y, z) = x^2+y^2+z^2$ be a function in $\mathbb{R}^3$. Then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 2z$.*

In one variable, we know that if $f'(x_0)$ exists, then f is continuous at x_0. For multiple variables, it is natural to consider the following question.

Question 1.2 *In $\mathbb{R}^n$, $n > 1$. Assume that $\frac{\partial f}{\partial x_1}(x), ..., \frac{\partial f}{\partial x_n}(x)$ exists for x near x_0, can one conclude $f(x)$ is continuous at x_0?*

Solution Answer is No. Here is a counter-example:

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

We know that f is not continuous at $(0, 0)$.

$$\frac{\partial f}{\partial x}(x, y) = \frac{y(x^2 + y^2) - 2x^2y}{(x^2 + y^2)^2} = \frac{y^2 - x^2y}{(x^2 + y^2)^2} \quad \text{if } (x, y) \neq (0, 0)$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{x^2 - y^2x}{(x^2 + y^2)^2} \quad \text{if } (x, y) \neq (0, 0)$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{x\to 0} \frac{f(x, 0) - f(0, 0)}{x} = 0 = \frac{\partial f}{\partial y}(0, 0).$$

So $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist for $x, y \in \mathbb{R}^2$. but $f(x, y)$ is not continuous at $(0, 0)$.

So we need some stronger notion of differentiability like one variable case.

Definition 1.3 *Let $f(x)$ be a function in $\mathbb{R}^n$, $x_0 \in \mathbb{R}^n$. We say that f is differentiable at x_0 if there is a vector, called $f'(x_0) \in \mathbb{R}^n$ such that*

$$\lim_{x\to x_0} \frac{|f(x) - f(x_0) - f'(x_0) \cdot (x - x_0)|}{\|x - x_0\|} = 0.$$

Definition 1.4 *Gradient of f at x_0 is a vector defined as:*

$$\nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), ..., \frac{\partial f}{\partial x_n}(x_0)\right)$$

Proposition 1.5 *If f is differentiable at x_0, then*
(1) All $\frac{\partial f}{\partial x_j}(x_0)$ exist and $f'(x_0) = \nabla f(x_0)$.
(2) f is continuous at x_0.

What is a natural sufficient condition to test if f is differentiable at x_0?

THEOREM 1.6 *If $\frac{\partial f}{\partial x_j}(x)$ exists and continuous at x_0 for all $1 \leq j \leq n$, then f is differentiable at x_0.*

Proof. We know $\frac{\partial f}{\partial x_j}(x)$ exists near $x = x_0$ and continuous at x_0. Choose $f'(x_0) = \nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0), ..., \frac{\partial f_n}{\partial x_n}(x_0)\right)$. Let

$$g(t) = f(tx + (1-t)x_0).$$

Then

$$\begin{aligned}
&\frac{|f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x)|}{\|x - x_0\|} \\
= \quad &\frac{|g(1) - g(0) - g'(0)|}{\|x - x_0\|} \\
= \quad &\frac{|g'(\theta) - g'(0)|}{\|x - x_0\|} \\
= \quad &\frac{|\nabla f(\theta x + (1-\theta)x_0) \cdot (x - x_0) - \nabla f(x_0) \cdot (x - x_0)|}{\|x - x_0\|} \\
\leq \quad &\frac{\|\nabla f(\theta x + (1-\theta)x_0) - \nabla f(x_0)\| \|(x - x_0)\|}{\|x - x_0\|} \\
= \quad &\|\nabla f(\theta x + (1-\theta)x_0) - \nabla f(x_0)\| \\
\to \quad &0 \text{ as } x \to x_0
\end{aligned}$$

since $\nabla f(x)$ is continuous at x_0. ▯

For higher order partial derivaties, we use:

$$\frac{\partial^2 f}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}\right), \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right), \quad \frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

where$\alpha = (\alpha_1, ..., \alpha_n), \alpha_j \geq 0, |\alpha| = \sum_{j=1}^n \alpha_j$.

Question 1.7 *Suppose $\frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ exist at (x_0, y_0). Does*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0)?$$

Solution Answer is No (in general). A counter example:

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

We show that $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist in $\mathbb{R}^2$, but that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

For $(x,y) \neq (0,0)$.

$$f(x,y) = xy - \frac{2y^2}{x^2+y^2} = -xy + \frac{2x^2}{x^2+y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = y + \frac{2y^2 \cdot 2x}{(x^2+y^2)^2} = y + \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = -x - \frac{4x^2 y}{(x^2+y^2)^2}$$

For $(x,y) = (0,0)$, we get $\frac{\partial f}{\partial x}(0,0) = 0$, $\frac{\partial f}{\partial y}(0,0) = 0$.

Now, for the second order derivatives:

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(-x - \frac{4x^2 y}{(x^2+y^2)^2}\right) \\ &= \frac{\partial}{\partial x}\left(-x - \frac{4y}{(x^2+y^2)^2} + \frac{4y^3}{(x^2+y^2)^2}\right) \\ &= 1 + \frac{8yx}{(x^2+y^2)^3} + \frac{16xy^3}{(x^2+y^2)^3} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(y + \frac{4xy^2}{(x^2+y^2)^2}\right) \\ &= \frac{\partial}{\partial y}\left[y + \frac{4x}{(x^2+y^2)} - \frac{4x^3}{(x^2+y^2)^2}\right] \\ &= 1 - \frac{8x^2}{(x^2+y^2)^2} + \frac{16x^4}{(x^2+y^2)^3} \end{aligned}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} = \lim_{x \to 0} \frac{-x-0}{x} = -1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = \lim_{x\to 0} \frac{y-0}{y} = 1$$

Therefore, $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Question: When $\frac{\partial^2 x}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$?

THEOREM 1.8 *If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}$ exist and $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at (x_0, y_0), then $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ exists and $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$*

In particular, if $f \in C^2(\mathbb{R}^n)$ then $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.

Proof. $h, k \in \mathbb{R}^n$. We consider

$$\nabla(f,h,k) = f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)$$

$$u(t) = f(x_0+h, t) - f(x_0, t)$$

Then

$$\begin{aligned} u(y_0+k) - u(y_0) &= (\frac{du}{dt}(y+\theta k)k = \frac{\partial f}{\partial y}(x_0+h, y_0+\theta k) - \frac{\partial f}{\partial y}(x_0, y_0+\theta k))k \\ &= \frac{\partial^2 f}{\partial x \partial y}(x_0+\theta_1 h, y_0+\theta k)hk \end{aligned}$$

Since $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at (x_0, y_0), $\exists\ \delta > 0$ such that if $|h| + |k| < \delta$, then

$$\left| \frac{\partial^2 f}{\partial x \partial y}(x_0+\theta_1 h, y_0+\theta k) - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right| < \epsilon.$$

Therefore $\left| \frac{\nabla(f,h,k)}{hk} - \frac{\partial^2 f}{\partial x \partial y}(x_0.y_0) \right| < \epsilon$ when $|h| + |k| \leq d_0$ (**). On the other hand,

$$\begin{aligned} \lim_{k\to 0}\lim_{h\to 0} \frac{\nabla(f,h,k)}{hk} &= \lim_{k\to 0} \frac{1}{k} \lim_{h\to 0} \frac{f(x_0+h, y_0+k) - f(x_0, y_0+k) - f(x_0+h, y_0) + f(x_0, y_0)}{h} \\ &= \lim_{k\to 0} \frac{1}{k} \left[\frac{\partial f}{\partial x}(x_0, y_0+k) - \frac{\partial f}{\partial x}(x_0, y_0) \right] \\ &= \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) \end{aligned}$$

As long as $\lim_{k\to 0}\lim_{h\to 0} \frac{\nabla(f,h,k)}{hk}$ exists. By (**), we have

$$\lim_{k\to 0}\lim_{h\to 0} \left[\frac{\nabla(f,h,k)}{hk} - \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \right] = 0.$$

Therefore, $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ exists and is equal to $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$. $\square$

Next, we consider a map $f : \mathbb{R}^n \to \mathbb{R}^m$.

$$f(x) = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; x_0 = \begin{bmatrix} x_1^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

We say that f is differentiable at x_0 if there is $m \times n$ matrix A such that

$$\lim_{x \to x_0} \frac{||f(x) - f(x_0) - A(x - x_0)||}{||x - x_0||} = 0.$$

We use the notation: $f'(x_0) = A$.

Proposition 1.9 *If f is differentiable at x_0, then $\frac{\partial f_i}{\partial x_j}(x_0)$ exists for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Moreover,*

$$f'(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_m}(x_0) \end{bmatrix}$$

The Jacobian of f is $\mathrm{Jac}(f) = \det f'(x)$.

Let $A = [a_{ij}]_{m \times n}$ matrix, we use $||A||^2$ to denote the largest eigenvalue of $A^*A (= A^T A$ when A is real). Then, $||AX|| \leq ||A|| ||X||$

THEOREM 1.10 *Let E be convex and if $f : E \to \mathbb{R}^m$ is differentiable and $||f'(x)|| \leq M$ on E, then $||f(x) - f(y)|| \leq M||x - y||$ for $x, y \in E$.*

Exercise

THEOREM 1.11 *(Chain Rule) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$ be differentiable. Then $g \circ f : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable and $(g \circ f)'(x) = g'(f(x))f'(x)$.*

1.2 Inverse function theorem and open mapping theorem

THEOREM 1.12 *(Inverse function theorem) Let E be a domain in $\mathbb{R}^n$, $f : E \to \mathbb{R}^n$ is C^1-mapping. Let $f'(x)$ be invertible at $x_0 \in E$. Then:*

(a) There is a $\delta > 0$ such that

(1) f is 1-1 on $B(x_0, \delta)$

(2) $f(B(x_0, \delta))$ is open.

(b) if $g : V = f(B(x_0, \delta)) \to B(x_0, \delta)$ is the inverse function of f, then $g \in C^1(V)$ and $g'(y) = (f'(g(y)))^{-1}$.

Corollary 1.13 *Open mapping theorem. If $f : E \to \mathbb{R}^n$ is C^1-mapping and $\det f'(x) \neq 0$ for all $x \in E$, then f is an open map, which maps open set to open set.*

Proof. Let $A = f'(x_0)$. Then A^{-1} exists. Let λ be a positive number such that $2\lambda||A^{-1}|| = 1$. For any $y \in \mathbb{R}^n$, define a map $\phi_y(x) = x + A^{-1}(y - f(x))$, then $\phi_y(x) = x$ if and only if $f(x) = y$. Since $f'(x)$ is continuous and $f'(x_0) = A$, there is $\delta > 0$ such that $||\phi'_y(x) - A|| < \lambda/n$. Then

$$\begin{aligned}
||\phi_y(x_1) - \phi_y(x_2)|| &\leq n \max ||\phi'_n(x)||||x - x_0|| \\
&= n \max ||I + A^{-1} f'(x)|| \\
&= n||A^{-1}|| \max ||A - f'(x)||||x - x_0|| \\
&\leq \lambda||A^{-1}||||x - x_0|| \\
&= 1/2||x_1 - x_2||, x_1, x_2 \in B(x_0, \delta)
\end{aligned}$$

Then, $\phi_y : B(x_0, \delta) \to \mathbb{R}^n$ is a contractive map and $\phi_y(x)$ has at most one fixed point. Then, $f(x) = y$ has at most one solution in $B(x_0, \delta)$.

Therefore, $f : B(x_0, \delta) \to \mathbb{R}^n$ is 1-1. Next, we need to show $f(B(x_0, \delta))$ is open. It is sufficient to prove: $x \in B(x_0, \delta), f(x) = y$, $y_0 = f(x_0)$, then $B(y_0, \lambda y) \subset f(B(x_0, \delta))$ where $\lambda = \delta - ||x - x_0|| > 0$. We consider $\phi_y(x) = x + A^{-1}(y - f(x))$. Since

1) $x_1 \in B(x_0, \delta)$ and $r > 0$ with $B(x_1, r) \subset B(x_0, \delta)$ then

$$y_1 = f(x_1) \in f(B(x_1, r)) \subset f(B(x_0, \delta)).$$

If $y \in B(y_1, \lambda r)$, then $y \in f(B(x_0, \lambda r))$ and

$$||\phi_y(x) - \phi_y(x_1)|| \leq (1/2)||x - x_1||.$$

$$\begin{aligned}
||\phi_y(x) - x_1|| &\leq ||\phi_y(x) - \phi_y(x_1)|| + ||\phi_y(x_1) - x_1|| \\
&\leq (1/2)||x - x_1|| + ||x_1 + A^{-1}(f(x_1) + y) - x_1|| \\
&= (1/2)||x - x_1|| + ||A^{-1}(f(x_1) - y|| \leq (1/2)r + ||A^{-1}||||y_1 - y|| \\
&\leq (1/2)r||A^{-1}||r \\
&\leq (1/2)r
\end{aligned}$$

This implies $\phi_y : \overline{B(x_1, r)} \to \overline{B(x_1, r)}$ is a contractive map. $||\phi_y(x_1) - \phi_y(x)|| \leq (1/2)||x_1 - x||$ for all $x, x_1 \in \overline{B(x_1, r)}$.

By Banach Fixed point theorem, there is a fixed point $x \in B(x', r)$ s.t. $\phi_y(x) = x \Leftrightarrow y = f(x)$, which implies $y \in f(B(x_1, r)) \subset f(B(x_0, r))$, therefore $f(B(x_0, \delta))$ is open. □

1.3 Implicit Function Theorem

THEOREM 1.14 *(Implicit Function Theorem) Let* $f = (f_1, \cdots, f_n) : \mathbb{R}^{n+m} \to \mathbb{R}^n$ *be a* C^1 *map with* $(a, b) \in \mathbb{R}^{n+m}$ *such that* $f(a, b) = 0$ *and* $f_x(a, b)$ *invertible. Then there is an open set* $U \subset \mathbb{R}^{n+m}$ *and an open set* $W \subset \mathbb{R}^m$

such that $(a,b) \in U$, $b \in W$ *and* $x = g(y), y \in W$ *satisfying* $(g(b), y) \in U$ *and* $f(g(y), y) = 0$, $y \in W$. *Moreover,* $g \in C^1(W)$ *and*

$$g'(y) = -(f_x(g(y), y))^{-1} f_y(g(y), y).$$

REMARK 1 *This means that* $f(x,y) = 0$ *determines a function* $x = g(y)$, $y \in W$. *Moreover,* $g'(y) = -(f_x)^{-1} f_y$. *Here*

$$f_x = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \quad f_y = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix}$$

Proof. We construct a map: $F = \begin{bmatrix} f(x,y) \\ y \end{bmatrix} : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, $F(a,b) = (0,b)$, F is C^1 map.

$$F'(x,y) = \begin{bmatrix} f_x & f_y \\ 0 & I_m \end{bmatrix}$$

det $F'(a,b) =$ det $f'_x(a,b) \neq 0$ by the inverse function theorem. There is $G : V \to U$, $(U = B((a,b), r), V = F(U))$ such that $G \circ F = I$ on U and $F \circ G = I$ on V. Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = F \circ G(x,y) \quad \text{and} \quad \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} f(G_1(0,y), G_2(0,y)) \\ G_2(0,y) \end{bmatrix}.$$

Therefore, $G_2(0,y) = y$. Let $x = g(y) =: G_1(0,y)$ and let

$$W = \{y : (0,y) \in V\}.$$

Then

$$f(g(y), y) = 0, \quad y \in W.$$

By the chain rule: $f_x(g(y), y) g'(y) + f_y(g(y), y) = 0$. This implies that

$$g'(y) = -(f_x(g(y), y))^{-1} f_y(g(y), y).$$

∎

EXAMPLE 2 *Let* $f = (f_1, f_2) : \mathbb{R}^{2+3} \to \mathbb{R}^2$ *be defined by*

$$f = \begin{bmatrix} 2e^{x_1} + x_2 y_1 - 4y_2 + 3 \\ x_2 \cos(x_1) - 6x_1 + 2y_1 - y_3 \end{bmatrix}$$

Let $a = (0,1)$, $b = (3,2,7)$. *Then*

$$f(a,b) = \begin{bmatrix} 2 + 3 - 4 \cdot 2 + 3 \\ 1 - 0 + 2 \cdot 3 - 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It is clear that f is C^1 map and

$$f_{(x,y)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{d y_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \end{bmatrix} = \begin{bmatrix} 2e^{x_1} & y_1 & x_2 & -4 & 0 \\ -x_2 \sin(x_1) - 6 & \cos(x_1) & 2 & 0 & -1 \end{bmatrix}$$

Then

$$f_x(a,b) = \begin{bmatrix} 2e^{x_1} & y_1 \\ -x_2 \sin(x_1) - 6 & \cos(x_1) \end{bmatrix} (a,b) = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}$$

which is non-singular since $det f_x(a,b) = 20 \neq 0$, by implicit function theorem $f(x,y) = 0$ determines uniformly a function $x = g(y)$ for $y \in W$ for some W with $b \in W$.

$$g'(y) = -(f_x(g(y), y))^{-1} f_y(g(y), y)$$

and

$$g'(b) = -\frac{1}{20} \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = -\frac{1}{20} \begin{bmatrix} -5 & -4 & 3 \\ 1 & -24 & -2 \end{bmatrix}$$

1.4 Exercise

1. Prove or disprove that there is a function $f(x)$ in $\mathbb{R}^n$ with $n > 1$ so that all first order partial derivaties $\frac{\partial f}{\partial x_j}(x)$ exist for all $x \in \mathbb{R}^n$ and $f(x)$ is continuous at 0, but $f(x)$ is not differentiable at $x = 0$.

2. Let $f(x) = Ax : \mathbb{R}^n \to \mathbb{R}^m$ with $A = [a_{ij}]$ is a $m \times n$ scalar matrix. Prove f is differentiable and find $f'(x)$.

3. Suppose f is differentiable on $[a,b]$, $f(a) = 0$ and there is $A > 0$ such that $|f'(x)| \leq A|f(x)|$ on $[a,b]$. Prove $f(x) = 0$ on $[a,b]$.

4. Let ϕ be a real function defined on $R = [a,b] \times [\alpha, \beta]$. A solution of the initial-value problem

$$y' = \phi(x,y), \quad y(a) = c \quad (\alpha \leq c \leq \beta).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1| \qquad (C)$$

whenever $(x, y_1), (x, y_2) \in R$.

5. Let $A = [a_{ij}]$ be $n \times n$ matrix. $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transform defined as $T_A(x) = Ax$. Let $\|T_A\| = \|A\| = \max\{\|Ax\| : \|x\| = 1\}$. Prove

$$\|A\|^2 \leq \sum_{i,j=1}^{n} a_{ij}^2 \leq n\|A\|^2$$

6. Suppose that $f(x)$ is a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ are bounded in E. Prove that f is continuous on E. Is f uniformly continuous on E?

7. Suppose that f is a real-valued differentiable function on an open set E in $\mathbb{R}^n$. If $x_0 \in E$ so that f attains its local maximum at x_0. Prove $f'(x_0) = 0$.

8. Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable maps. Then the product map $g(x) \cdot f(x) : \mathbb{R}^n \to \mathbb{R}$ is differentiable and

$$(g \cdot f)'(x) = g(x)f'(x) + f(x)g'(x)$$

by viewing $f(x)$ and $g(x)$ as row vectors in $\mathbb{R}^m$.

9. Define $f(0,0) = 0$ and $f(x,y) = x^3/(x^2+y^2)$ if $(x,y) \neq (0,0)$. Prove that $\frac{\partial f(x)}{\partial x}$ and $\frac{\partial f}{\partial y}$ exists on $\mathbb{R}^2$ and bounded on $\mathbb{R}^2$, but f is not differentiable at $(0,0)$.

10. Let $f(x) = (x_1^2 - x_2^2, x_2^2 - x_3^2, x_1 + x_2 - x_3) : \mathbb{R}^3 \to \mathbb{R}^3$ and $g(y_1, y_2, y_3) = (\sin(y_1 - y_2), \cos(y_1 + y_3), y_1 y_2 y_3) : \mathbb{R}^3 \to \mathbb{R}^3$. Find $(g \circ f)'(x)$.

11. Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a mapping defined as follows:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1^2 - x_2 + \sin(x_1 - x_3), \\ f_2(x_1, x_2, x_3) &= x_2^2 - x_3 + (x_1 + x_2 + x_3)^3, \\ f_3(x_1, x_2, x_3) &= x_1 + x_2 - x_3 \end{aligned}$$

Prove that there is $\delta > 0$ so that
(i) $f : B(0,\delta) \to V = f(B(0,\delta))$ is one-to-one and onto;
(ii) V is an open neighborhood of 0 in $\mathbb{R}^n$
(iii) If $g : V \to B(0,\delta)$ is the inverse of f, then find $g'(0)$.

12. (a) State the implicit function theorem;
(b) Let $f(x) = (f_1(x), f_2(x)) : \mathbb{R}^4 \to \mathbb{R}^2$ be a C^1 map defined as follows:

$$\begin{aligned} f_1(x) &= x_1 - x_2^2 + e^{x_1 - x_3 - x_4} - 1 \\ f_2(x) &= x_1^1 + x_2 + \sin(x_1 + x_2 - x_3 - x_4) \end{aligned}$$

Show that
(i) $f(0) = 0$
(ii) In a neighborhood E of 0 in $\mathbb{R}^2$, there is a C^1 map $(x_1, x_2) = (g_1(x_3, x_4), g_2(x_3, x_4))$ so that

$$f(g_1(x_3, x_4), g_2(x_3, x_4), x_3, x_4) = 0, \quad \text{in } E$$

(iii) Find $g'(x_3, x_4)$