Real Analysis

Qualifying Exam

WEDNESDAY, SEPTEMBER 24, 2014
1:00 pm – 3:30 pm

Problem	1	2	3	4	5	6	Σ
Points							

Student's name:

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Problem 1.

Let A be the collection of all subsets of $\mathbb{R}$ that consist of exactly 5 points. Find the σ-algebra of sets generated by A.

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Problem 2.

Assume that $f \in L^1((0,1))$ is a non-negative real valued function satisfying $\int_{[0,1]} f(x)\, dx = 1$. Show that

$$\int_{[0,1]} \frac{1}{f(x)}\, dx \geq 1.$$

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Problem 3.

Denote

$$E = \left\{ x \in [0,1] \,\middle|\, \text{there exist infinitely many } p, q \in \mathbb{N} \text{ such that } |x - \frac{p}{q}| \leq \frac{1}{q^3} \right\}.$$

Show that $m(E) = 0$.

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Problem 4.

Assume that $\eta \in L^1(\mathbb{R})$ is a non-negative function satisfying $\int_{\mathbb{R}} \eta \, dx = 1$. Show that for any $f \in L^1(\mathbb{R})$,

$$||f * \eta||_{L^1(\mathbb{R})} \leq ||f||_{L^1(\mathbb{R})}.$$

Here

$$(f * \eta)(x) = \int_{\mathbb{R}} f(x-y)\eta(y) \, dy.$$

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Problem 5.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period one. Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)\cos^2(2\pi x)dx = \frac{1}{2}\int_0^1 f(x)dx.$$

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Problem 6.

Let $x = 0.n_1n_2\ldots$ be the decimal representation of $x \in [0,1]$. Compute with justification $\int_{[0,1]} f(x)dx$, where

1. $f(x) = \max_{i\in\mathbb{N}} |n_i - n_{i+1}|$
2. $f(x) = \max_{i\in\mathbb{N}} \sum_{k=i}^{\infty} \frac{n_k}{2^{k-i}}$