COMPLEX ANALYSIS

Qualifying Exam

Thursday, September 17, 2015 — 1:00 pm - 3:30 pm, MSTB 122

Problem	1	2	3	4	5	6	7	8	Σ
Points									

Student's name:

Problem 1.

Prove that the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges uniformly in the unit disc $|z| \leq 1$. Does the series obtained by term-by-term differentiation converge uniformly in the unit disc? Explain your answer.

Problem 2.

Let f be an entire function and suppose that there exists a bounded sequence $\{a_n\}$ of real numbers such that $f(a_n)$ is real for all $n \in \mathbb{N}$. Prove that $f(x)$ is real for all real x.

Problem 3.

Evaluate

$$\int_0^\infty \frac{\sin ax}{x(x^2+b^2)}dx, \quad a, b > 0.$$

Problem 4.

How many roots does the equation $z + e^{-z} = a$, $a \in \mathbb{R}, a > 1$, have in the right half plane?

Problem 5.

Find a real valued function $u(z)$ that is continuous in the closed disc $\overline{D(0, R)}$ (that is, closed disc centered at 0 of radius $R > 0$) and harmonic in $D(0, R)$, and satisfies

$$u(Re^{i\theta}) = \frac{1}{2}(1 + \cos^3 \theta), \quad \theta \in [0, 2\pi).$$

Problem 6.

Consider a non-constant polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots a_1 z + a_0, \;\; n \geq 1, \;\; a_0, a_n \neq 0,$$

and set $B = \max_{0 \leq j \leq n-1} |a_j|$, $C = \max_{1 \leq j \leq n} |a_j|$. Prove that all roots of the polynomial P lie inside the annulus $r \leq |z| \leq R$, where

$$r = \frac{1}{1 + \frac{C}{|a_0|}}, \;\; R = 1 + \frac{B}{|a_n|}.$$

Problem 7.

TRUE or FALSE: The family $\mathcal{F}$ of functions holomorphic in a unit disc with power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that satisfy $|a_n| \leq n^{2015}$ is normal.

Problem 8.

Set $U_1 = \{1 < |z| < 2\}$ and $U_2 = \{0 < |z| < 1\}$.

a) Show that homeomorphism $f : U_1 \to U_2$ given by $f(re^{i\theta}) = (r-1)e^{i\theta}$ is not a conformal mapping;

b) Does there exist a conformal mapping $g : U_1 \to U_2$?