1. There are five separate statements listed below. For each, say whether it is true or false. For true statements cite an appropriate theorem or give a justification. For false statements provide counterexamples.

   If E_n is a sequence of measurable sets in a measure space (X, μ) such that $\mu(E_n) \to 0$, and $f : X \to \mathbb{R}$ is measurable, is it true that you can conclude that $\int_{E_n} f d\mu \to 0$ if

   (a) $f \geq 0$.

   (b) E_n are nested and decreasing.

   (c) $X = \mathbb{R}^n$ and f is continuous.

   (d) $X = \mathbb{R}^n$ and f is locally integrable.

   (e) $f \in L^1(X, \mu)$.

2. Consider Lebesgue measure on the real line.

   a) Let $X \subset \mathbb{R}$ be a measurable subset, and consider a sequence of real measurable functions f_n. Suppose that $f_n \to 0$ a.e. in X. If $\int_X f_n^4 \, dx \leq 1$, is it true that

$$\lim_{n\to\infty} \int_X |f_n| dx = 0,$$

   if

   a) $X = [0, 1]$;
   b) $X = \mathbb{R}$.

   Prove or give a counterexample.

3. Let f be a measurable function on $[0, 1]$. Assume that for any $t \geq 0$, $m\{|f| \geq t\} \leq \frac{1}{t^2}$. Show that $f \in L^1([0, 1])$.

4. Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space with $\mu(X) = \infty$.

   (a) Prove that there exist $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$, disjoint, such that $1 \leq \mu(B_n) < \infty$ for all $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} B_n$.

   (b) Prove that there exists an $\mathcal{A}$-measurable real-valued function f on X such that $f \notin L^1$ but $f \in L^p$ for all $p \in (1, \infty)$.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that

   a) $L^1(X, \mathcal{A}, \mu) \cap L^\infty(X, \mathcal{A}, \mu) = \bigcap_{p\in[1,\infty]} L^p(X, \mathcal{A}, \mu)$.

   b) It is not always true that $L^1(X, \mathcal{A}, \mu) \cup L^\infty(X, \mathcal{A}, \mu) \supset \bigcap_{p\in(1,\infty)} L^p(X, \mathcal{A}, \mu)$.

6. Let U be open in $\mathbb{R}^2$. Show that U is equal to the union of pairwise disjoint open balls and a set of Lebesgue measure zero.

   Hint: First show there are disjoint open balls $B_i \subset U$ with $m(\cup_{i=1}^{\infty} B_i) > \frac{m(U)}{5}$.