Real Analysis Qualifying Exam 2016 Spring

Your Name: ____________________________________

INSTRUCTIONS: *Complete all problems if possible. Use only one side of each sheet. Do at most one problem on each page. Write your name on every page. Justify your answers. Where appropriate, state without proof results that you use in your solutions.*

Problem	Point Value	Points Received
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

Your Name: ____________________________________

Problem 1: Assume that $f \in L^1([0,1])$. Compute

$$\lim_{k \to +\infty} \int_{[0,1]} |f|^{\frac{1}{k}} \, dx.$$

Justify your answer.

Your Name: ___________________________________

Problem 2: Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functions on $[0,1]$ and $0 \leq f_n \leq 1$ a.e. Assume that

$$\lim_{n\to+\infty} \int_{[0,1]} f_n g\, dx = \int_{[0,1]} f g\, dx$$

for some $f \in L^1([0,1])$ and any $g \in C([0,1])$. Prove that $0 \leq f \leq 1$ a.e.

Your Name: ________________________________

Problem 3: Let $f, g \in L^2(\mathbb{R}, \mathcal{M}_L, \mu_L)$. Show that $f * g$ is a continuous function on $\mathbb{R}$ vanishing at infinity, that is, $f * g \in C(\mathbb{R})$ and $\lim_{|x| \to \infty} (f * g)(x) = 0$.

Your Name: ___________________________________

Problem 4: Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and let $p_1 \in (1, \infty]$. Let $\{f_n\}_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $L^{p_1}(X, \mathcal{A}, \mu)$, that is, $\{f_n\}_{n\in\mathbb{N}} \subset L^{p_1}(X, \mathcal{A}, \mu)$ and $\sup_{n\in\mathbb{N}} ||f_n||_{L^{p_1}} < \infty$. Suppose $f = \lim_{n\to\infty} f_n$ exists μ-a.e. Prove that $f \in L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1]$ and $f_n \to f$ in $L^p(X, \mathcal{A}, \mu)$ for all $p \in [1, p_1)$.

Your Name: ______________________________

Problem 5: Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0, \infty)$ be $\mathcal{A}$-measurable. Consider the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu_L)$, where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$ and μ_L is Lebesgue measure, and form the product measure space $(X \times \mathbb{R}, \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}}), \mu \times \mu_L)$. Define $E \subset X \times \mathbb{R}$ by
$(x, y) \in E \iff y \in [0, f(x))$. Prove that $E \in \sigma(\mathcal{A} \times \mathcal{B}_{\mathbb{R}})$ and
$(\mu \times \mu_L)(E) = \int_X f \, d\mu$.

Your Name: ________________________________

Problem 6: Let $f \in L^1(\mathbb{R})$, and let $a_1, \ldots, a_k \in \mathbb{R}$ and $b_1, \ldots, b_k \in \mathbb{R} \setminus \{0\}$. Assume that the vectors $\frac{a_j}{b_j}$ are all distinct. Determine

$$\lim_{t \to \infty} \int \left| \sum_{j=1}^{k} f(b_j x + t a_j) \right| dx.$$