ALGEBRA COMPREHENSIVE EXAM

FALL 2012

NAME	
SIGNATURE	

Time: 2 hours and 30 minutes. *Each problem is 10 points. Closed book and closed notes. Please explain all your answers.*

If you are giving an example or a counter-example please explain why it satisfies the condition of the problem.

Problem	1	2	3	4	5	6	7	8	9	10
Score										

Problem 1

Let H, K be two subgroups of a finite group G, such that $H \cap K = \{e\}$ and $|H| \cdot |K| = |G|$. Does it follow that $G \simeq H \times K$ as groups? Prove or give a counter-example.

Problem 2

Compute the number of conjugacy classes in the dihedral group D_{2n} with 2n elements, and also the order of the commutator subgroup D'_{2n}.

Problem 3

Show that a simple group with 168 elements has at least 14 elements of order 3.

Problem 4

Give an example of a commutative ring with 1 that has exactly two maximal ideals.

Problem 5

Let F be a field and $f(x)$ a polynomial in $F[x]$. Suppose that $R = F[x]/(f)$ is an integral domain. Show that R is in fact a field.

Problem 6

Assume that a linear map $\phi : V \to W$ of vector spaces over a field is *not* surjective. Show that there exists a vector space U and a pair of *distinct* linear maps ψ_1, ψ_2 from W to U; such that $\psi_1 \circ \phi = \psi_2 \circ \phi$ as linear maps from V to U.

Problem 7

Classify, up to conjugation, all 4×4 real matrices with minimial polynomial

$$(x^2 + 4)(x - 1).$$

Problem 8

Show that a normal operator N on a finite dimensional Hermitian vector space can be represented in the form $N = A+B$ where $A^* = A$, $B^* = -B$ and $AB = BA$.

Problem 9

Suppose that $\alpha, \beta \in \mathbb{C}$ have minimal polynomials over $\mathbb{Q}$ of degrees 2 and 3, respectively. Can $\alpha + \beta$ have minimal polynomial over $\mathbb{Q}$ of degree 5? Give an example or prove that it is not possible.

Problem 10

Let p be a prime and $\mathbb{F}_p \subset \mathbb{F}_{p^n}$ a degree $n > 1$ extension of finite fields. Consider the Frobenius auotmorphism $\Phi : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ sending α to α^p. Show that Φ is $\mathbb{F}_p$-linear, that its minimal polynomial $m_\Phi(T)$ has degree n, and then compute the minimal polynomial. (HINT: $\sum a_i \Phi^i = 0$ means that every α is a root of $\sum a_i x^{p^i}$)