ALGEBRA QUALIFYING EXAM

June 16, 2014

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators. *Notation:* Let $\mathbb{Z}$ denote the integers.

1. (a) Is $\mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/48\mathbb{Z}$ isomorphic to $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/60\mathbb{Z}$? Explain.
   (b) Let $N = 561$ and $b \in \mathbb{Z}$ such that $\gcd(b, N) = 1$. Prove that $b^{N-1} \equiv 1 \pmod{N}$.
2. Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a cyclic quartic extension of $\mathbb{Q}$, i.e., is a Galois extension of degree 4 with cyclic Galois group.
3. A commutative ring with unity R is called a *local ring* if it has a unique maximal ideal M. Prove that if R is a local ring, then $R^\times = R - M$. (Here $R^\times$ is the group of units of R.)
4. Let F be a field. Prove that the ring $F[x^2, x^3]$ is not a unique factorization domain.
5. Prove that no group of order 351 is simple.
6. Answer "True" or "False", and FULLY JUSTIFY your answer with a proof or counterexample:
   If G is a group, and H and K are subgroups of G such that $H \lhd K \lhd G$, then $H \lhd G$.
7. Answer "True" or "False", and FULLY JUSTIFY your answer with a proof or counterexample:
   Suppose that R be a commutative ring with 1, and M is an ideal of R. Then M is a maximal ideal of R if and only if R/M is a field.
8. Prove that two 3×3 matrices over a field are similar if and only if they have the same characteristic and minimal polynomials.
9. Give definitions for each of the following:
   (a) group
   (b) ring,
   (c) integral domain,
   (d) module,
   (e) homomorphism of modules.
10. Prove that every finite field is perfect, i.e., that every finite extension of a finite field is separable.