1. (a) Is $\mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/48\mathbb{Z}$ isomorphic to $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/60\mathbb{Z}$? Explain.

(b) Let $N = 561$ and $b \in \mathbb{Z}$ such that $\gcd(b, N) = 1$. Prove that $b^{N-1} \equiv 1 \pmod{N}$.

2. Show that $\mathbb{Q}(\sqrt{2} + \sqrt{2})$ is a cyclic quartic extension of $\mathbb{Q}$, i.e., is a Galois extension of degree 4 with cyclic Galois group.

3. A commutative ring with unity $R$ is called a local ring if it has a unique maximal ideal $M$. Prove that if $R$ is a local ring, then $R^\times = R - M$. (Here $R^\times$ is the group of units of $R$.)

4. Let $F$ be a field. Prove that the ring $F[x^2, x^3]$ is not a unique factorization domain.

5. Prove that no group of order 351 is simple.

6. Answer “True” or “False”, and FULLY JUSTIFY your answer with a proof or counterexample:
   - If $G$ is a group, and $H$ and $K$ are subgroups of $G$ such that $H \lhd K \lhd G$, then $H \lhd G$.

7. Answer “True” or “False”, and FULLY JUSTIFY your answer with a proof or counterexample:
   - Suppose that $R$ be a commutative ring with 1, and $M$ is an ideal of $R$. Then $M$ is a maximal ideal of $R$ if and only if $R/M$ is a field.

8. Prove that two $3 \times 3$ matrices over a field are similar if and only if they have the same characteristic and minimal polynomials.

9. Give definitions for each of the following:
   - (a) group
   - (b) ring,
   - (c) integral domain,
   - (d) module,
   - (e) homomorphism of modules.

10. Prove that every finite field is perfect, i.e., that every finite extension of a finite field is separable.