APPENDIX
OPEN AND SOLVED PROBLEMS

This list of problems, mainly from the first edition, is supplemented at the sign ‡ with information about solutions, or partial solutions, achieved since the first edition. The questions in section F are new to this edition.

AT THE SIGN ✠ WE ADD NEW INFORMATION SINCE THE PUBLICATION OF THE BOOK.

A. Almost free modules.
1. Is it true that for any ring which is not left-perfect, $\text{Inc}'(R) = \text{Inc}(\mathbb{Z})$?
2. If we give the general definition of κ-separable, are direct summands of κ-separable groups κ-separable? [Note that in the case where $|A| = \kappa$, if A is a counterexample then $\Gamma(A) = 1$.]
3. Is it true that if κ-free does not imply κ^+-free, then κ-separable does not imply κ^+-separable? [Note Exercises VII.13–17.]

‡*Shelah 1996 answers this in the affirmative. See §XV.4.*

4. (Kaplansky test problem for $\aleph_1$-separable groups) Suppose A, B are countable torsion-free groups so that A is not isomorphic to B, but $A \oplus A \cong B \oplus B$. Are there $\aleph_1$-separable groups G, H (of cardinality $\aleph_1$) so that $G \oplus G \cong H \oplus H$, the quotient type of G is A, and the quotient type of H is B? [cf. Thomé 1988 and 19??a and b]

‡*Eklof-Shelah 1998 proves, in ZFC, strong negative answers to the Kaplansky test problems for $\aleph_1$ -groups of cardinality $\aleph_1$. See §XV.2. The precise question asked here is not answered.*

5. Find a large cardinal equiconsistent with "there is a κ, so that κ-free implies free". [Shelah has shown that the first such cardinal has some large cardinal properties.]
6. (Droste) Is it consistent that $\text{Inc}(\mathbb{Z})$ is countable?

‡*Magidor-Shelah 1994 contains the known closure properties of* Inc($\mathbb{Z}$)*; see* § *VII.5; the question of whether* Inc($\mathbb{Z}$) *can be countable remains open.*

7. For any variety $\mathcal{V}$, is λ in the essentially non-free incompactness spectrum if and only if there is some n such that (CP_n) holds and there is a λ-free family of countable sets based on a λ-system of height n? [See VII.3A.17.]

8. Does the incompactness spectrum of groups equal that of abelian groups?

‡*A special case is handled in the thesis of C. Bitton (Hebrew University, 1998) but the general problem remains open.*

B. Structure of Ext.

1. Find a combinatorial principle equivalent to the existence of a non-free W-group (of arbitrary cardinality)? [See XII.3].

‡*Solved in Eklof-Shelah 1994. See XIII.2.11.*

2. Is it consistent that the class of W-groups of cardinality $\aleph_1$ is exactly the class of strongly $\aleph_1$-free groups of cardinality $\aleph_1$?

‡*The answer is no. See Eklof-Mekler-Shelah 1992.*

3. If we have, say, all the uniformization results that can be deduced from MA + ¬CH, then is every strongly $\aleph_1$-free (every Shelah) group of cardinality $\aleph_1$ a W-group?

4. If every strongly $\aleph_1$-free group of cardinality $\aleph_1$ is a W-group, are they also all $\aleph_1$-coseparable?

‡*There is a partial result in Eklof-Mekler-Shelah 1992. See XIII.2.10.*

5. Does strongly $\aleph_1$-free plus $\aleph_1$-coseparable imply $\aleph_1$-separable (for groups of cardinality $\aleph_1$)?

‡*Answered in the negative by Eklof-Shelah. (The published version in "Abelian groups and Modules", Marcel Dekker Lecture Notes in Math no. 182 has an error; corrected version available from the first author or the second author's web-site.)*

6. Is it consistent that there are filtration-equivalent $\aleph_1$-separable groups of cardinality $\aleph_1$ such that one is a W-group but the other is not?

✠ Shelah-Strüngmann (ShSm855, arXiv:math.LO/0612241) show that the answer is yes.

C. Endomorphism rings.

1. Characterize the endomorphism rings of separable groups modulo the idea of small endomorphisms.

D. Dual groups.

1. If one predual of a dual group is a dual group, must they all be dual groups?

‡ *Schlitt 1999 has an example of a dual group A which has a* ***maximal*** *predual which is a dual group, and another* ***maximal*** *predual which is not a dual group. See XVII.1.11.*

2. Investigate the foundation rank of dual groups.

3. Suppose A_0, A_1, A_2, A_3 is a sequence of groups where $A_{i+1} = A_i^*$; if B is the subgroup annihilated by ker ρ_3, is B a maximal predual of A_1?

‡*Schlitt 1999 shows that the answer is no. See XVII.1.10.*

4. If A is a dual group, which groups can appear as $A^{**}/\sigma[A]$? [Recall from XVII.1.7 that any such group is a dual group]

‡*Schlitt 1993 shows that any dual group can so appear. See XI.5.1.*

5. Is it provable in ZFC that every $\aleph_1$-separable group of cardinality $\aleph_1$ is a dual group?

6. Is there a reflexive group of ω-measurable cardinality?

✠Shelah has shown that the answer is yes; see SH904, arXiv:math/0703493; ALG. UNIV. v. 63 (2010), 351–366

7. Is it provable in ZFC that there exists A such that A^* and A^{**} are both slender?

✠The problem as stated has trivial solutions, e.g. $\mathbb{Q}$. Since the problem was inspired by Eklof-Mekler-Shelah 1987, a better, as yet unsolved, problem is:

Is it provable in ZFC that there exists A such that A, A^* and A^{**} are all ω_1-free, not finitely-generated, and slender?

8. Is it provable in ZFC that there exists $A \subseteq \mathbf{Z}^\omega$ which is non-reflexive? [follows from CH] Even assuming CH, is there such an A which is strongly non-reflexive?

‡*Eda-Kamo-Ohta 1993 shows that there is a dual group $A \subseteq \mathbb{Z}^\omega$ which is strongly non-reflexive, without assuming CH.*

9. (Huber) Is it provable in ZFC that every W-group (of cardinality $\aleph_1$) is reflexive?

‡*Eklof-Shelah 2001a shows that the answer is no; in fact, it is*

consistent that there is a W-group of cardinality $\aleph_1$ whose dual group is free. Eklof-Shelah 200?b proves the same result is consistent with GCH.

10. Is there a **Z**-chain of strongly non-reflexive dual groups i.e., groups A_n ($n \in \mathbf{Z}$) such that for all n, $A_n^* = A_{n+1}$ and A_n is not isomorphic to A_{n+2}? (And for other partial orders.)

‡*Ohta 1996 proves that it is consistent that there is a Z-chain of strongly non-reflexive dual groups, and has proved in ZFC that there is such a chain of order type the opposite of ω.*

11. Is there a group A such that A^{**} is not H^{***} for any H?

12. If A is a dual group of infinite rank, is $A \cong A \oplus \mathbf{Z}$?

‡*Göbel-Shelah 2001b constructs, assuming $\diamondsuit$, a reflexive, hence dual, group A for which the answer is no. See XVII.5.6. See also Göbel-Shelah 2001c and 2001d. It remains open whether the conclusion is provable in ZFC.*

E. Others.

1. Is Reid_μ closed under direct summands?

2. Investigate dependence/independence among the arrows in IV.2, VII.4, especially: is it consistent that every W-group is free but not every hereditarily-separable group is a W-group?

‡ *Eklof-Mekler-Shelah 1994 proves that it is consistent that every W-group is free but there are non-free hereditarily separable groups. See XIII.4.4.*

3. Does every $\aleph_1$-separable group of cardinality $\aleph_1$ have a coherent system of projections?

‡*Eklof-Mekler-Shelah 1993 shows that it is consistent with either CH or $\neg$CH that there is an $\aleph_1$-separable group of cardinality $\aleph_1$ with no coherent system of projections.*

F. More.

1. Is it consistent that there are $\aleph_1$-free splitters of cardinality $\aleph_1$ which are not free? [See §XVI.3.]

2. Is every countably generated Baer module, over an arbitrary integral domain, *projective*? [See §XII.3.]

✠ANGELERI HÜGEL-BAZZONI-HERBERA (TRANS. AMER. MATH. SOC. V. 360 (2008), 2409-2421) PROVE THAT THEY ARE ALL PROJECTIVE.

3. Can Theorem XVI.2.9 be extended (in ZFC) to classes $\mathcal{P}$ which are not classes of pure-injective modules? [In Eklof-Trlifaj 2000 it is extended to classes of cotorsion modules over Dedekind domains.]

✠ Eklof-Shelah-Trlifaj (J. Algebra v. 277 (2004), 572-578) shows that the result in Eklof-Trlifaj 2000 is best possible in ZFC for Dedekind domains with countable spectrum.

4. Is it provable in ZFC + GCH that if κ is minimal such that there are non-free κ-free groups of cardinality κ and every κ-free group of cardinality κ is *Whitehead*, then κ is inaccessible? [See XIII.3.14; there is some evidence for this in Shelah 200?b.]

✠See SH914 (arXiv: 0708.1908) for a related problem.

5. Is Proposition XVI.1.13 provable in ZFC?

✠ Bazzoni-Eklof-Trlifaj (Bull. LMS, v. 37 (2005), 683-696) shows that the answer is yes.

6. (Trlifaj) Is there a slender p.i.d. (or an integral domain) such that the Whitehead modules, of arbitrary cardinality, are exactly the $\aleph_1$-free modules? [cf. Eklof-Shelah 200?d]

Last updated March 6,2012
Comments are welcome; please email peklof@math.uci.edu