Algebra Qualifying Exam, Winter 2000

Student Name and ID Number:

This exam contains 14 problems. Do as many problems as you can. A complete and correct solution of ten (10) or more problems is a PH.D. pass. A complete and correct solution of seven (7) or more problems is a Master pass. Although some partial credits might be given, complete solutions are much prefered.

1. Let G be a finite group acting on a finite set S. For each element $g \in G$, let $S^g = \{s \in S | g(s) = s\}$ be the subset of elements of S fixed by g. For $s \in S$, let $G_s = \{g \in G | g(s) = s\}$ be the stablizer of s.
   a. Prove the formula $\sum_{s \in S} |G_s| = \sum_{g \in G} |S^g|$ (Hint: consider the set of pairs (g, s) satisfying $g(s) = s$).
   b. Prove Burnside's formula: $|G| \times$ (number of orbits) $= \sum_{g \in G} |S^g|$.

2. Let T be a linear operator of an n-dimensional vector space V over a field F (not necessarily algebraically closed), where n is a positive integer. Show that there is a basis $\vec{e}$ of V such that the matrix A of T with respect to $\vec{e}$ has at least $n(n-1)/2$ zero entries.

3. Let V_n be the vector space of complex polynomials $f(x)$ of degree at most n, where n is a positive integer. Let D be the matrix of the linear differential operator d/dx acting on V_n with respect to some basis of V_n. Prove that D is not diagonalizable.

4. Let H be a normal subgroup of prime order p in a finite group G. Suppose that p is the smallest prime dividing $|G|$. Prove that H is in the center $Z(G)$ of G (Hint: let $ghg^{-1} = h^i$ and show that one can take $i = 1$).

5. A subgroup G of $(\mathbb{R}^2, +)$ is called discrete if the topological closure of G in $\mathbb{R}^2$ has no limiting points in it. Show that any discrete non-cyclic subgroup G of $(\mathbb{R}^2, +)$ is a lattice in $\mathbb{R}^2$, i.e., G is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

6. Show that the alternating A_6 has no subgroup of index 5.

7. Let A be an matrix in O_n with determinant -1. Show that -1 is an eigenvalue of A. Give an example showing that the same result is false without the determinant -1 condition.

8. Show that the compact group SU_2 has exactly 7 complex representations of dimension 5 and write down all the 7 representations in terms of the irreducible representations of SU_2 (Hint: use the fact that SU_2 has exactly one irreducible representation of degree n for each positive integer n).

9. Let R be the ring $\mathbb{Z}[\sqrt{-5}]$.

a. Sow that R is not a UFD.

b. Factor the principal ideal (6) into a product of prime ideals in the ring R.

10. Determine the direct sum structure of the abelian group A generated by $\{x, y, z\}$ with the following three relations:

$$7x + 5y + 2z = 0,\ 10x + 8y + 2z = 0, 13x + 11y + 2z = 0$$

11. Let $\mathbb{F}_q$ be the finite field of q elements with characteristic p. Let $n > d$ be positive integers. Prove that the generalized Fermat equation

$$x_1^d + x_2^d + \cdots + x_n^d = 0$$

has a non-trivial solution with coordinates in $\mathbb{F}_q$. (Hint: first find the sum $\sum_{x \in \mathbb{F}_q} x^k$ for non-negative integers k.)

12. Let R_1, R_2 be polynomial rings in finite number of variables. Show that the product ring $R_1 \times R_2$ is a Noetherian ring.

13. Determine the Galois group of the polynomial $x^p - 2$ over $\mathbb{Q}$, where p is an odd prime number.

14. Let n be a positive integer. Prove that the polynomial $x^{4n} + 8x + 13$ is irreducible over $\mathbb{Q}$. (Hint: make a change of variable and use the Eisenstein criteria.)