ALGEBRA QUALIFYING EXAMINATION

September, 2001

There are 11 questions, but you are not expected to answer all of them. Do as many problems as you can. We prefer complete solutions of a few problems to many partial solutions. 60 total points is sufficient to pass at the Master's level. 75 total points is sufficient to pass at the Ph.D. level The number of points for each section of a problem (or for the whole problem, if it is not subdivided into sections) is given in parenthesis. Even if the problem is not subdivided into sections, partial credit may be given. You have 150 minutes. GOOD LUCK!

Notation: If R is ring, R^* denotes the multiplictive group of units of R. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ denote the complex, real, rational and integer numbers rpectively. $\mathbb{F}_q$ denotes the field with q elements.

1. **Action of $GL_3(\mathbb{F}_2)$**
   Let V be the vector space of dimension 3 over the field with 2 elements $\mathbb{F}_2$. Let $GL_3(\mathbb{F}_2)$ be the 3×3 invertible matrices over the integers modulo 2. Let G be the group consisting of the pairs (v, g) with $v \in V$ and $g \in GL_3(\mathbb{F}_2)$ using the multiplication for which the product of (v_1, g_1) times (v_2, g_2) is $(v_1 + g_1(v_2), g_1g_2)$.
   a. **(3 points)** Show there is no group in G properly between G and $H = \{(0, g) \mid g \in GL_3(\mathbb{F}_2)$.
   b. **(2 points)** Let G act on the eight elements of V by (v_1, g_1) maps $v \in V$ to $v_1 + g_1(v)$.
   Show that this action identifies G with a subgroup of S_8 (i.e. it induces an *injective* map $G \to S_8$).
   c. **(5 points)** Since there are eight elements in V, part a) gives G as a subgroup of S_8. Show G is actually a subgroup of A_8. Hint: Show that the elements (v, I_3), $v \in V$ and I_3 the 3×3 identity matrix are in A_8. Then use Jordan canonical form to show the group H in part a) is also in A_8.

2. **Groups of order p^2 and p^3**
   Let G be a finite group. Let p be a prime number.
   a.(**5 points)** Assume that G has p^2 elements. Use that G as a nontrivial center to show that G is an abelian group.
   b. (**5 points)** Find a nonabelian group G having order 3^3. Hint: Look in the upper trianglular matrices over $\mathbb{F}_3$.

3. **Groups of order pq**
   a. (**5 points)** Prove that every group of order 15 is abelian. Hint: Use Sylow's Theorem (state precisely what you use about it).
   b. **(5 points)** Suppose p and q are two distinct primes. For which p and q is there exactly one group of order pq?

4. **The Galois Group of a degree 5 polynomial**
   Let $f(x)$ be an irreducibe degree p polynomial over $\mathbb{Q}$ with exactly $p - 2$ real roots where p is a prime. Regard the Galois group G_f of $f(x)$ as a subgroup of S_p through its action on the roots of f.
   a. (**3 points)** Show G_f contains a 2-cycle of S_p.
   b. (**3 points)** Show $G_f = S_p$. Hint: Use that the irreducibility of f implies that G_f is transitive subgroup. Explain why p being a prime now implies G_f contains a p-cycle.
   c. (**4 points)** Let $f(x) = x^5 - 9x + 2$. Using a. and b. show that $G_f = S_5$.

5. **Representations of D_4**

This problem decribes all irreducible representations of D_4, the dihedral group of order 8. Recall: The number of irreducible represenations of a finite group G is the same as the number of conjugacy classes in G. Further, the sums of squares of the degrees of these representations is the same as $|G|$

a. (**4 points**) Give a representative of each conjugacy class in D_4. Find the center and the commutator subgroup of D_4.

b. (**3 points**) Describe the standard 2-dimensional representation of D_4.

c. (**3 points**) List all irreducible representations of D_4.

6. **Symmetric and antisymmmtric matrices.**

Recall that matrix X is symmetric if $X^t = X$ and is antisymmetric if $X^t = -X$. Let S denote the space of all real antisymmetric $n \times n$ matrices.

a. (**2 points**) Find the dimesnsion of S over R.

b. (**2 points**) For a fixed $n \times n$ matrix A let $F_A(X) = A^tX + XA$. Show that F_A maps S to S.

c. (**6 points**) Compute the rank of F_A (as a linear operator on S), and compute the eigenvalues of F_A as linear combinations of the eigenvalues of A.

7. **Representations of S_3**

The group S_3 acts on $\{1, 2, 3\}$. In this probem we consider the vector space representation of S_3 related to this action

a. (**2 points**) Give a homomorphism of S_3 into GL(3,C) with trivial kernel, (corresponding to an action of S_3 on $V = C^3$).

b. (**3 points**) Find a 1-dimensional subspce V_1 of V that is left fixed by S_3.

c. (**5 points**) Find generators of a 2-dimensional subspace V_2, preserved by S_3, such that V is a direct sum of V_1 and V_2.

8. **Multiplicative group of a field**

(**10 points**) Prove that any finite subgroup G of the multiplicative group K^* of a field K is cyclic.

9. **Maximal Ideals in a quotient ring**

a. (**3 points**) Find all maximal ideals of the ring $T = Q[X]/(X^2 - 4)$

b. (**3 points**) Find all maximal ideals of the ring $S = Q[X]/(X^2 - 1)$

a. (**4 points**) Express T (part a.) as a direct sum of fields

10. **The 2-power map in characteristic 2**

a. (**5 points**) Find an irreducible degree 3 poynomial $f(x)$ in $F_2[x]$. Explain why $F_8 = F_2[\alpha]$ where α is any root of $f(x)$.

a. (**5 points**) $B = \{1, \alpha, \alpha^2\}$ is a basis for F_8 over F_2 (where α is as in part a.). Let $T_2 : F_8 \to F_8$ be the linear map $T_2(\beta) = \beta^2$ for all β in F_8. Compute the matrix of T_2 relative to the basis B.

11. **Algebraic Integers in an imaginary quadratic field.**

Consider the field $Q[(-15)^{1/2}]$

a. (**4 points**) Describe the ring A of algebraic integers in $Q[(-15)^{1/2}]$

b. (**3 points**) Does the prime number 3 remain prime in A? Explain.

c. (**3 points**) Does the prime 2 ramify in A?