ALGEBRA QUALIFYING EXAM

June 14, 2010

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{F}_q$ denote the finite field with q elements. Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers. Let $\mathbb{C}$ denote the complex numbers.

1. Classify all groups of order 44, up to isomorphism. Make clear which of them are abelian.
2. Answer true or false for each of the following, and briefly explain your answer:
   (a) The group $\{(1), (12)\}$ is a normal subgroup of S_5.
   (b) The center $Z(G)$ of any group G is a normal subgroup of G.
   (c) For any group G, the map $\theta(g) = g^2$ from G to itself is a homomorphism.
3. Let R be the ring $\mathbb{Z}[\sqrt{-5}]$.
   (a) Show that R is not a UFD.
   (b) Factor the principal ideal (6) into a product of prime ideals in the ring R.
4. Let n be a positive integer. Prove that the polynomial $f(x) = x^{2^n} + 8x + 13$ is irreducible over $\mathbb{Q}$.
5. Let T be a linear operator of an n-dimensional vector space V over a field F. Assume that T is nilpotent, i.e., there is some positive integer k such that $T^k = 0$. Show that $T^n = 0$.
6. Construct the character table of the dihedral group of order 8.
7. For each of the following two rings, determine whether or not it is a field:
   (a) $\mathbb{F}_2[x]/(x^3 + x + 1)$,
   (b) $\mathbb{F}_3[x]/(x^3 + x + 1)$.
8. List exactly one representative from each similarity class of matrices $A \in \mathrm{GL}_2(\mathbb{C})$ such that A is similar to A^{-1}.
9. Determine the splitting field over $\mathbb{Q}$ of $x^4 - 2$. Then determine the Galois group over $\mathbb{Q}$ of $x^4 - 2$, both as an abstract group and as a set of automorphisms. Give the lattice of subgroups and the lattice of subfields. Make clear which subfield is the fixed field of which subgroup.
10. Find one representative from each similarity class of matrices over $\mathbb{Q}$ whose characteristic polynomial is $x^5 + x^3$.