ALGEBRA QUALIFYING EXAM

September, 2010[1]

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators.

Notation: Let $\mathbb{F}_q$ denote the finite field with q elements. Let $\mathbb{Z}$ denote the integers. Let $\mathbb{Q}$ denote the rational numbers.

1. Let K/F be an algebraic extension of fields and let R be a ring such that $F \subseteq R \subseteq K$. Prove that R is a field.
2. Show that if G is a group of order $(35)^3$, then G is not a simple group.
3. Determine the splitting field of $x^5 - 2$ over the finite field $\mathbb{F}_3$. Then determine the Galois group over $\mathbb{F}_3$ of $x^5 - 2$, both as an abstract group and as a set of automorphisms.
4. Let χ be the character of a d-dimensional complex representation ρ of a finite group G. Prove that $|\chi(g)| \le d$ for all $g \in G$, and that if $|\chi(g)| = d$, then $\rho(g) = \zeta I$ for some root of unity ζ depending on g.
5. Find the Galois group over $\mathbb{Q}$ of $x^3 + 4x + 2$, as an abstract group.
6. Let $\mathbb{R}$ denote the real numbers and let R be the ring $\mathbb{R}^n$, where n is a positive integer. Let $I \subseteq R$ be an ideal. Prove that I is maximal if and only if there is a coordinate i such that $I = \{\langle r_1, \ldots, r_n\rangle \in R \mid r_i = 0\}$.
7. Give definitions for each of the following:
   (a) ring,
   (b) module,
   (c) ring homomorphism,
   (d) module homomorphism.
8. Suppose that A is a finite abelian group, S is the Sylow p-subgroup S of A, and p^k is the order of S. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to S.
9. Give an example of an extension of fields that is not separable. Compute its separable and inseparable degrees. (Fully justify your answer.)
10. (a) How many similarity classes of matrices over $\mathbb{Q}$ have characteristic polynomial $(x^4 - 1)(x^2 - 1)$?
    (b) Find one representative from each similarity class.
    (c) For each similarity class, give the minimal polynomial of the matrices in that class.

[1] Transcribed by Joshua Hill, 2014-03-29.