Algebra Qualifying Exam, September 2013

Instructions: JUSTIFY YOUR ANSWERS. LABEL YOUR ANSWERS CLEARLY. Each of the 10 questions is worth 10 points. Do as many problems as you can, as completely as you can. The exam is two and one-half hours. No notes, books, or calculators. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the ring of integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively.

1. (a) Show that the order of $\mathrm{GL}_2(\mathbb{F}_p)$ is $(p^2-1)(p^2-p)$.
   (b) Give an example of a p-Sylow subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$.
   (c) How many p-Sylow subgroups does $\mathrm{GL}_2(\mathbb{F}_p)$ have?

2. (a) Describe all automorphisms of the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. How many are there?
   (b) Describe all automorphisms of the ring $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. How many are there?

3. Suppose R is a commutative ring with 1, and $x \in R$ lies in every maximal ideal of R. Prove that $1+x$ is a unit in R.

4. Let M be an $n \times n$ matrix over a field F.
   (a) If F has characteristic zero, show that M is nilpotent (i.e., $M^k = 0$ for some integer $k > 0$) if and only if $\mathrm{Tr}(M^i) = 0$ for $1 \leq i \leq n$.
   (b) Give an example to show that the same statement is not true if the field F has positive characteristic $p > 0$.

5. Prove that for any field F and any positive integer n, the group $\mu_n(F) = \{x \in F | x^n = 1\}$ is a cyclic group.

6. Let $f(x) = x^2 + x + 2 \in \mathbb{F}_5[x]$.
   (a) Prove that $f(x)$ is irreducible in $\mathbb{F}_5[x]$.
   (b) Explain why $f(x)$ divides the polynomial $x^{25} - x$ in $\mathbb{F}_5[x]$.
   (c) How many irreducible quadratic polynomials are there in $\mathbb{F}_5[x]$?

7. Let E be the splitting field of $x^{21} - 1$ over $\mathbb{Q}$.
   (a) What is the degree $[E : \mathbb{Q}]$?
   (b) How many subfields does E have?

8. For relatively prime positive integers m and n, show that
$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0.$$

9. Consider the extension of fields $\mathbb{R}(T) \subset \mathbb{R}(T^{1/4})$, where T is an indeterminate.
   (a) Is $\mathbb{R}(T^{1/4})/\mathbb{R}(T)$ Galois? Why or why not?
   (b) Find all intermediate fields F such that $\mathbb{R}(T) \subseteq F \subseteq \mathbb{R}(T^{1/4})$, and prove you have found them all.

10. Let G be a finite group acting on a finite set S. Let $\mathbb{C}[S]$ be the abstract vector space over $\mathbb{C}$ with basis S. Let χ be the character of the corresponding representation of G on $\mathbb{C}[S]$.
   (a) Show that for $\sigma \in G$, the value $\chi(\sigma)$ is the number of fixed points of σ in S.
   (b) Show that the inner product $\langle \chi, 1_G \rangle$ is the number of G-orbits in S, where the inner product is given by $\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_1(\sigma)\chi_2(\sigma^{-1})$.