SOME USES OF SET THEORY IN ALGEBRA

Stanford Logic Seminar
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Plan

I. The Whitehead Problem – early history

II. Compactness and Incompactness

III. Deconstruction

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P. Eklof and A. Mekler, **Almost Free Modules**, *North-Holland (1990)*; Preface begins:

"The modern era in set-theoretic methods in algebra can be said to have begun on July 11, 1973 when Saharon Shelah borrowed László Fuchs' *Infinite Abelian Groups* from the Hebrew University library. Soon thereafter, he showed that Whitehead's Problem — to which many talented mathematicians had devoted much creative energy — was not solvable in ordinary set theory (ZFC)."

> *"One day I have come and see the second volume of László; its colour was attractive green. I take it and ask myself isn't everything known on [abelian groups]... I start to read each linearly; after reading about two thirds of the first volume I move to the second volume and read the first third. I mark the problems (I think six) which attract me—combination of being stressed by László, seem to me I have a chance, and how nice the problem look"*
>
> *"I have thought the most important is to build indecomposable abelian groups in every cardinality. I thought the independence of Whitehead's problem will be looked on suspiciously. As you know abelian group theorists thought differently." [communication from Shelah]*

I. The Whitehead Problem

"module" means left R-module, where R is a ring

"group" means abelian group, i.e. $\mathbb{Z}$-module

A module is **free** if it has a basis
or, equivalently, is isomorphic to a direct sum of copies of R.

Fact: A subgroup of a free group is free.
(This is not true for modules in general.)

The Whitehead Problem

Is every Whitehead group (of arbitrary cardinality) free?
(Ehrenfeucht 1955)

Fact: Every free group is a Whitehead group.

Classic result: Every countable Whitehead group is free. (Stein 1951; Ehrenfeucht 1955)

Fact: A subgroup of a Whitehead group is a Whitehead group.

Hence, if A is a Whitehead group of cardinality $\aleph_1$, then every countable subgroup of A is free, i.e., A is $\aleph_1$-free.

S. Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel J. Math **18** (1974)

Theorem (Shelah)

(1) Assuming V = L, every Whitehead group of cardinality $\aleph_1$ is free;

(2) Assuming MA + ¬ CH, there is a Whitehead group of cardinality $\aleph_1$ which is not free.

The proof of (1) requires $\diamondsuit_{\omega_1}(S)$ for *every* stationary subset S of $\aleph_1$.

In Math. Reviews, a review of L. Fuchs' **Infinite Abelian Groups, vol II** states:

> *Since Volume II was written, S. Shelah [Israel J. Math. 18 (1974), 243–256] has shown that the statement "Every W-group of cardinal $\aleph_1$ is free" is independent of ZFC (Paul Hill has shown that GCH implies that every W-group of cardinality $\aleph_\Omega$ is free and J. Rotman has an easy proof that CH implies that every W-group is free).*

The mathematical assertions in parenthesis are wrong, as Shelah later showed:

Devlin-Shelah (c. 1977): weak CH ($2^{\aleph_0} < 2^{\aleph_1}$) implies "weak diamond" implies that Whitehead groups are $L_{\infty\omega_1}$-free.

But

Shelah (c. 1977): It is consistent with ZFC + GCH that there are Whitehead groups of cardinality $\aleph_1$ which are not free.

IIa. Compactness

Shelah's original paper proves:

Theorem. Assuming V = L, if κ is a regular uncountable cardinal and every Whitehead group of cardinality $< \kappa$ is free, then every Whitehead group of cardinality κ is free.

(*The proof uses* $\diamondsuit_\kappa(S)$ *for* every *stationary subset* S *of* κ*.*)

An inductive argument shows that every Whitehead group of cardinality $\aleph_n$ is free for all $n \in \omega$.

To go further one needs a result for singular cardinals.

Shelah's Singular Compactness Theorem

Let λ be a singular cardinal and A a group [abelian or non-abelian] of cardinality λ such that every subgroup of cardinality $< \lambda$ is free. Then A is free.

Hence: assuming V = L, **every** Whitehead group is free.

Shelah's Singular Compactness Theorem was much more general.

Let $\mathcal{F}$ be a class of modules; the members of $\mathcal{F}$ will be called "$\mathcal{F}$-free".

Shelah's Singular Compactness Theorem–more general version

Suppose λ is a singular cardinal and M is a module of cardinality λ such that for all $\kappa < \lambda$, *enough* submodules of cardinality κ are $\mathcal{F}$-free; then M is $\mathcal{F}$-free.

This is a template for a theorem. A specific example will be given later.

IIb. Incompactness

Question: For which regular uncountable cardinals κ is there an abelian group of cardinality κ which is not free, but such that every subgroup of cardinality $< \kappa$ is free? Such a κ will be called **incompact**.

FACTS: (1) A weakly compact cardinal is not incompact.

(2) Assuming V = L, a regular uncountable κ is incompact only if it is not weakly compact.

(3) [Magidor-Shelah] Every regular uncountable cardinal $< \aleph_{\omega^2}$ is incompact. Moreover, if $\aleph_\alpha$ and $\aleph_\beta$ are incompact, then so are $\aleph_{\alpha+1}$ and $\aleph_{\alpha+\aleph_\beta+1}$.

(4)[Magidor-Shelah] It is consistent with ZFC + GCH (assuming the consistency of certain large cardinals) that $\aleph_{\omega^2+1}$ is not incompact.

(5) [Shelah] An uncountable cardinal κ is incompact if and only if there is a family of size κ of countable sets which does not have a transversal (a one-one choice function) but every subfamily of size $< \kappa$ does.

III. Deconstructibility

Let $\mathcal{A}$ be a class of modules.

Definitions

$\mathcal{A}$ is μ-deconstructible if every module in $\mathcal{A}$ is the union of a chain of submodules $\{A_\alpha : \alpha < \sigma\}$ which are members of $\mathcal{A}$ such that:
(1) $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ for all limit ordinals $\beta < \sigma$; and
(2) for all $\alpha < \sigma$, $A_{\alpha+1}/A_\alpha \in \mathcal{A}$ and has cardinality $< \mu$.

$\mathcal{A}$ is deconstructible if it is μ-deconstructible for some μ.

We will restrict to classes of the form
$\mathcal{A} = {}^\perp\mathcal{B} = \{A \mid \mathrm{Ext}^1(A, B) = 0 \text{ for all } B \in \mathcal{B}\}$ for some set or class $\mathcal{B}$.
Recall: $\mathrm{Ext}^1(A, B) = 0$ iff every short exact sequence

$$0 \to B \to M \to A \to 0$$

splits, i.e., up to isomorphism the only one is

$$0 \to B \to B \oplus A \to A \to 0$$

If $\mathcal{B} = \{\mathbb{Z}\}$, then $\mathcal{A}$ is the class of **Whitehead groups**.

Lemma.

Fix $\mathcal{A}$ and μ. If $\mathcal{F}$ is the class of μ-deconstructible members of $\mathcal{A}$, then the Singular Compactness Theorem applies to the notion of $\mathcal{F}$-free (for sufficiently large singular cardinals); i.e.,
If λ is a sufficiently large singular cardinal and $M \in \mathcal{A}$ has cardinality λ and enough submodules of cardinality $< \lambda$ are μ-deconstructible, then M is μ-deconstructible.

Theorem.

Assuming $V = L$, the class of Whitehead groups is $\aleph_1$-deconstructible.

Corollary.

Assuming V = L, every Whitehead group is free

The proof uses the homological fact:
If $A = \bigcup_{\alpha<\sigma} A_\alpha$ and $A_0 = 0$ and $A_{\alpha+1}/A_\alpha$ are free for all $\alpha < \sigma$, then A is free.

Baer modules

Let R be an integral domain. Let $\mathcal{T}$ be the class of torsion-free R-modules.
Let $\mathcal{A} = {}^{\perp}\mathcal{T} = \{A \mid \mathrm{Ext}^1(A, B) = 0 \text{ for all } B \in \mathcal{T}\}$.
Say A is a **Baer module** if it belongs to $\mathcal{A}$.

Question (Kaplansky): are all Baer modules over an arbitrary ID projective (i.e., a direct summand of a free module)?

Theorem. (Eklof-Fuchs-Shelah 1990)

The class of Baer modules is $\aleph_1$-deconstructible.

Question: Are the countably-generated Baer modules projective?

Theorem. (Angeleri Hugel-Bazzoni-Herbera 2005)

Every countably-generated Baer module over an arbitrary ID is projective.
Hence, *every* Baer module is projective.

Definition

A module T is *n-tilting* if:

(1) T has proj. dim. $\leq n$;

(2) $\mathrm{Ext}^i(A, B) = 0$ for all $i \geq 2$ and all $A \in {}^{\perp}(\{T\}^{\perp}), B \in \{T\}^{\perp}$; and

(3) $\{T\}^{\perp}$ is closed under direct sums.

Tilting modules

A tilting module has *finite type* (resp. *countable type*) if there is a set S of finitely-presented modules (resp. countably-presented modules) such that $S^{\perp} = \{T\}^{\perp}$.

Theorems

1. (Bazzoni-E-Trlifaj 2003) All 1-tilting modules are of countable type.
2. (Bazzoni-Herbera 2005) All 1-tilting modules are of finite type.
3. (Šťovíček-Trlifaj 2005) All n-tilting modules are of countable type.
4. (Bazzoni-Šťovíček 2005) All n-tilting modules are of finite type.

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