MT Outlined Solns

1a) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\cos t} \overset{\text{Hint}}{=} \frac{-4\sin t\cancel{\cos t}}{\cancel{\cos t}} = \boxed{-4\sin t}$

$\frac{dy}{dx} = 0$ at $t = \pm\pi, 0$ then plug in $\rightsquigarrow$ $\boxed{(x,y) = (0,1)}$ for all 3 times.

1b) $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{-4\cos t}{\cos t} = \boxed{-4}$ $\rightsquigarrow$ $\frac{d^2y}{dx^2} < 0$ for all time

so $\boxed{\text{concave down}}$.

2a) $\hat{u} = \frac{\langle 3,-5,4\rangle}{\sqrt{9+25+16}} = \frac{1}{\sqrt{50}}\langle 3,-5,4\rangle = \boxed{\frac{1}{5\sqrt{2}}\langle 3,-5,4\rangle}$

2b) Trick: $\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$ already unitized ✓ $\rightsquigarrow$ use $\boxed{\pm\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle}$

Alt: solve $\langle a,b\rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle = 0$ $\rightsquigarrow$ $\frac{a}{2} + \frac{b\sqrt{3}}{2} = 0$ $\rightsquigarrow$ $a = -\sqrt{3}\,b$

so $\langle -\sqrt{3}b, b\rangle$ is $\perp$ ✓

$\rightsquigarrow$ unitize and take $\pm$ use $\pm\frac{\langle -\sqrt{3}b, b\rangle}{\sqrt{3b^2+b^2}}$

$\rightsquigarrow$ Answer: $\boxed{\pm\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}\rangle}$

2c) It's $\text{Proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a} = \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}}{(1)^2}\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle$

$$= \boxed{\left(\frac{1+\sqrt{3}}{2}\right)\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\rangle}$$

3a) $\vec{AB} = \langle -1, 1, 0 \rangle$; $\vec{AC} = \langle 1, 0, -1 \rangle$.

Recall: $|\vec{AB} \times \vec{AC}|$ = Area of Parallelogram & $\text{Area}_{\text{Triangle}} = \frac{1}{2}\text{Area}_{\text{para.}}$

Here. $\vec{AB} \times \vec{AC} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = -\hat{i} - \hat{j} - \hat{k}$.

so, $A_{tri} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{\sqrt{1+1+1}}{2} = \boxed{\frac{\sqrt{3}}{2}}$

3b) Need to check volume of parallelepiped w/ legs $\vec{AB}, \vec{AC}, \vec{AD}$ will work ✓ (HW)

Alt Soln: From A, B, C, $\rightsquigarrow$ $\vec{AB} = \langle 0, 1, 1 \rangle$

(construct plane w/ pts A, B, C) $\vec{AC} = \langle -1, 1, 0 \rangle$.

$\hookrightarrow$ $\vec{n} = \vec{AB} \times \vec{AC} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = -\hat{i} - \hat{j} + \hat{k}$

so plane eqn (use pt A) is $-(x-1) - y + z = 0$.

* This plane contains A, B, C (check the pts satisfy this plane eqn)

Does it contain D? $\rightsquigarrow$ Plug in $D = (0, 0, 1)$ into the eqn.

$-(-1) - 0 + 1 = 2 \neq 0$.

No! Doesn't satisfy the plane eqn $\Rightarrow$ Not Coplanar ✓

4a) Plane's $\vec{n} = \langle 1, 0, -2 \rangle$ is thus the direction of line ✓

$\Rightarrow$ Use $\vec{n}$ and pt $(-1, 2, 1)$, $\vec{r}(t) = \langle -1, 2, 1 \rangle + t \langle 1, 0, -2 \rangle$

or, in components. $x = -1 + t$; $y = 2$; $z = 1 - 2t$

4b) First write $L_2 \rightsquigarrow x = 2 - 3s,\ y = s,\ z = -2-5s.$

Then solve for L_1 to equal L_2: $2-3s = 2t+3 \rightsquigarrow 2+3t = 2t+3$

$s = -t$ $\Rightarrow \boxed{t=1,\ s=-1}$

$-2-5s = -2t+5$

This matches x, y-comp ✓

Check/Plug in to z: $-2-5(-1) = 3$

$-2(1)+5 = 3$ ✓ so z-comp matches ✓

Thus, all components match ✓ Plug t into $L_1 \rightsquigarrow$ Pt is $\boxed{(5,-1,3)}$

Check with $L_2 \rightsquigarrow (5,-1,3)$ ✓

4c) We got the pt in (b) ✓

For $\vec{n}$, need to cross line directions $\vec{d}_1 = \langle 2,-1,-2\rangle$, $\vec{d}_2 = \langle -3,1,-5\rangle$

so $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ -3 & 1 & -5 \end{bmatrix} = 7\hat{i} + 16\hat{j} - \hat{k}$

Thus, Plane Eqn: $\boxed{7(x-5)+16(y+1)-(z-3) = 0}$

5a) $x^2+z^2=y^2 \sim \boxed{\text{Cone}}$

5b) $(x-1)^2+y^2+1 = 2z$

$\boxed{\text{Elliptic Paraboloid}}$

6a) Note $z = -y^2 \rightsquigarrow$ Let $\boxed{y=t}$ so $\boxed{z=-t^2}$

then $3x = +2yz \Rightarrow \boxed{x = \frac{2}{3}(-t^3)}$

Gives curve of intersection ✓

Between Pts $\left(\frac{2}{3},-1,-1\right) \rightsquigarrow$ from y, $\boxed{t=-1}$

and $(0,0,0) \rightsquigarrow \boxed{t=0}$, we can use the formula.

Also, $\boxed{\vec{r}\,'(t) = \langle -2t^2, 1, -2t\rangle}$, derivative of $\vec{r}(t)$

So, $L = \int_{-1}^{0} \sqrt{4t^4+1+4t^2}\,dt = \int_{-1}^{0} \sqrt{(2t^2+1)^2}\,dt$

$$= \int_{-1}^{0} (2t^2+1)\,dt = \frac{2t^3}{3} + t \Big|_{-1}^{0} = \frac{2}{3}+1 = \boxed{\frac{5}{3}}$$

6b) At $\left(-\frac{2}{3}, 1, -1\right) \rightsquigarrow \underline{t=1}$, so $\underline{\vec{r}\,'(1) = \langle -2, 1, -2 \rangle}$.

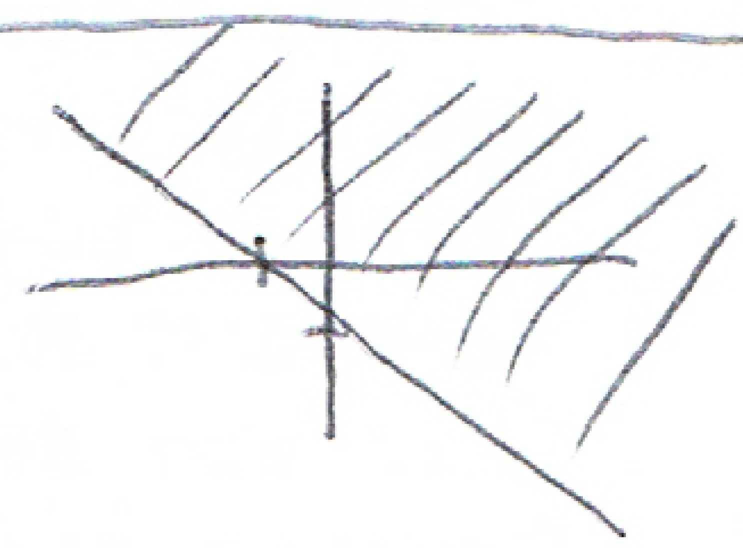
Also need $\vec{r}\,''(t) = \langle -4t, 0, -2 \rangle \rightsquigarrow \underline{\vec{r}\,''(1) = \langle -4, 0, -2 \rangle}$

Thus $K(t)$ at $t=1$: $K(1) = \dfrac{|\vec{r}\,'(1) \times \vec{r}\,''(1)|}{|\vec{r}\,'(1)|^3}$

We need: $\vec{r}\,'(1) \times \vec{r}\,''(1) = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -2 \\ -4 & 0 & -2 \end{bmatrix} = -2\hat{i} - 4\hat{j} + 4\hat{k}$.

so, $K(1) = \dfrac{\sqrt{4+16+16}}{(\sqrt{9})^3} = \dfrac{6}{27} = \boxed{\dfrac{2}{9}}$

7a) Need $x+y+1 \geq 0 \rightsquigarrow \boxed{y \geq -1-x}$, like



ie. $D = \{ (x,y) \in \mathbb{R}^2 \text{ such that } y \geq -1-x \}$.

7b) If $f \equiv k$, $\sqrt{x+y+1} = k$

So, $\underline{k \geq 0}$ only, and $\rightsquigarrow$

equivalently $\boxed{\begin{array}{l} x+y = k^2-1, \\ y = (k^2-1)-x \end{array}}$