Ch. 2.5 Worksheet - Math 3D April 27th

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Chapter 2.5 - Nonhomogeneous (Inhomogeneous) Equations

Summary: Only for 2nd order equations, like $y'' + ay' + by = f(x)$ or $x^2y'' + axy' + by = f(x)$.
The general solution is of the form $y = y_c + y_p$ where y_c solves the homogeneous equation (Right side = 0) and y_p solves specifically for the Right side's inhomogeneity (the $f(x)$).
They are called the Complementary and Particular pieces.

There are two methods for inhomogeneous equations:
1) Variation of Parameters: Harder to solve but "always" works.
2) Undetermined Coefficients: Easier to solve but needs a good guess.

For both, the 1st step is to find the form of $y_c(x) = C_1y_1(x) + C_2y_2(x)$. From there:

1) For Variation of Parameters: ** Make sure the coefficient of y'' is a 1 !! **
- Solve for the functions u_1', u_2' from the system

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = f(x) \end{cases}$$

- Integrate to find u_1, u_2. If allowed, it is okay to use definite integrals.
- The particular solution is given by $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$.
- Then $y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x)$. If we have initial conditions, we can solve for C_1 and C_2.
* When solving for C_1, C_2, don't forget there is a $y_p(x)$ there, too! *

2) For Undetermined Coefficients: Make an educated "guess" for a general form of $y_p(x)$.
Then, plug y_p into the differential equation and solve for the specific form of y_p.
Here are some guesses and some rules of guessing. They follow from the idea of taking derivatives:

. If $f(x)$ is a polynomial of degree k: Guess $y_p(x) = A_0 + A_1x + .. + A_kx^k$.
. If $f(x) = \sin(\omega x)$ and/or $\cos(\omega x)$: Guess $y_p(x) = A_1\sin(\omega x) + A_2\cos(\omega x)$. Include both!
. If $f(x) = e^{\lambda x}$: Guess $y_p(x) = Ae^{\lambda x}$.
*Rule 1: If $f(x)$ has a piece of the Complementary solution, rescale the guess by x^d where d is the multiplicity of the *root* that yields the complementary piece.
*Rule 2: If $f(x)$ is a sum or product of these types of functions, y_p is similarly a sum or product of the types of guesses.
. Solve for the constants by making $y_p'' + ay_p' + by_p = f(x)$ hold. Or if we are working with the Cauchy Euler equation, $x^2y'' + axy' + by = f(x)$ needs to hold.
. After we've solved for y_p, then $y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x)$. Again, we can solve for C_1, C_2 if we have initial conditions. Don't forget there's a $y_p(x)$ there!

If you can't see the guess for undetermined coefficients, or don't want to do undetermined coefficients, you have to use Variation of Parameters. Variation of Parameters is mainly useful if $f(x)$ is not very nice and we don't have any good guesses (e.g. $\frac{1}{x}$ from lecture, or $\sin(x^2)$ in the homework).

Some Exercises

1. Consider $y'' - 5y' + 6y = e^{3x}$. Find a particular solution with Variation of Parameters. If you were to use Undetermined coefficients, what would you guess for the form of y_p?

2. (2.5.102 tweaked)
If $y'' - 2y' + y = e^x + x^3$, find a particular solution to the equation.

3. What would be a good guess for y_p with Undetermined Coefficients for the following equations:
i. $y'' - 2y' + y = e^{3x} + x^2 + x$.
ii. $y'' + 4y = \cos 2x + \sin x$.
iii. $y^{(5)} - y^{(4)} = x^2$.
For practice, try to solve for y_p for some of them; maybe use simpler right hand sides, like just e^{3x} in (i) or just $\cos(2x)$ in (ii) for example.

Cauchy Euler Equation

These are equations like in Problem 2.1.5 in the homework. They are of the form

$$ax^2y'' + bxy' + cy = 0 \quad \text{or} \quad f(x) \quad \text{if inhomogeneous.}$$

(The scale a can also be divided out through all the terms).
To get the homogeneous (complementary) solution, assume that $y = x^r$ (instead of $y = e^{rx}$ for constant coefficient cases!) and solve for r. Doing this with the Cauchy Euler Equation gets

$$x^r\left[ar(r-1) + br + c\right] = 0$$

$$r^2 + r(b-a) + c = 0$$

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4c}}{2}.$$

We have to consider the same cases as before so I just wanted to tabulate the solution types.

1) Distinct roots λ_1, λ_2: Then our two solutions and hence general solution are

$$y = C_1x^{\lambda_1} + C_2x^{\lambda_2}.$$

2) One repeated root λ (mult. 2): Then our two solutions and hence general solution are

$$y = C_1x^{\lambda} + C_2x^{\lambda}\ln(x).$$

3) Complex pair, $\alpha \pm i\beta$: Our two solutions and hence general solution are

$$y = C_1x^{\alpha}\cos(\beta\ln x) + C_2x^{\alpha}\sin(\beta\ln x).$$

For examples, see the one in the practice midterm (#5) and Problem 2.1.5 in the homework.