Math 3D Quiz 1 Morning - April 13th

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Show all of your work. *There is a question on the back side.*

1. Consider the equation

$$y' = \frac{e^{-y}}{x}, \quad y(1) = 1.$$

(a) [3pts] Prove that this has a unique solution. (Hint: Use Picard's Theorem).

• Here, $f(x,y) = \frac{e^{-y}}{x}$. +1 $x_0 = 1,\ y_0 = 1$

+1 (i) f is well defined & continuous away from $x=0$, and fine for all y.
Initial $x_0 = y_0 = 1$, so it's fine $\Rightarrow$ f is continuous at I.C. $(1,1)$.

+1 (ii) $\frac{\partial f}{\partial y} = -\frac{e^{-y}}{x}$ is the same function (with (-) sign) so it's continuous at I.C. $(1,1)$.

Thus, Picard's Thm $\Rightarrow$ $\boxed{\text{A solution exists and is unique}}$ ✓

(b) [7pts] Find the explicit solution to the equation.

↳ Separate, $\frac{dy}{dx} = \frac{e^{-y}}{x} \Rightarrow e^{y}\,dy = \frac{dx}{x}$. +1

• Integrate, $e^{y} = \ln|x| + c$. +2

• Solve C and branch of $|x|$: From I.C. $y(1) = 1$,

$e = \ln|1| + c$; $\boxed{c = e}$ +1 and $|x| = |1|$ initially; positive branch. +1

Thus, $e^{y} = \ln(x) + e \Rightarrow \boxed{y = \ln(\ln(x) + e)}$ +1

For Domain: Need • $x > 0$ for $\ln(x)$
• $\ln(x) + e > 0$ for $\ln(\ln(x) + e)$

* $\ln(x) + e > 0 \Rightarrow \ln(x) > -e$, $x > e^{-e}$. +1

This is stricter than $x > 0$ so our domain is $\boxed{x > e^{-e}}$

2. (a) [8pts] Using an integration factor, find the solution to

$$x' + 3t^2 x = t^2, \quad x(0) = 1.$$

1st way, get general soln + solve for C:

Define $R(t) = e^{\int 3t^2 dt} = e^{t^3}$. +2

Multiply to both sides,

$$\frac{d}{dt}\left[e^{t^3} \cdot x\right] = t^2 e^{t^3} \quad +1$$

Integrate, $e^{t^3} \cdot x = \int t^2 e^{t^3} dt$

$$\overset{u=t^3}{=} \int \frac{e^u}{3} du = \frac{e^u}{3} + C$$

$$= \frac{e^{t^3}}{3} + C.$$

So, $x = e^{-t^3}\left[C + \frac{e^{t^3}}{3}\right]$ +3

I.C. for C: $1 = e^0\left[C + \frac{e^0}{3}\right]$, $1 = C + \frac{1}{3}$,

$\boxed{C = \frac{2}{3}}$ +2 $\Rightarrow$ $\boxed{x = e^{-t^3}\left[\frac{2}{3} + \frac{e^{t^3}}{3}\right]}$

SAME

2nd way with formula:

- $R_{sp}(t) = e^{\int_0^t 3t^2 dt}$

$$= e^{t^3 - 0} = e^{t^3}. \quad +2$$

Then, from formula,

+2

$$x = \frac{1}{e^{t^3}}\left[1 + \int_0^t \hat{t}^2 e^{\hat{t}^3} d\hat{t}\right]$$

$$\overset{u=\hat{t}^3}{=} e^{-t^3}\left[1 + \int_0^{t^3} \frac{e^u}{3} du\right]$$

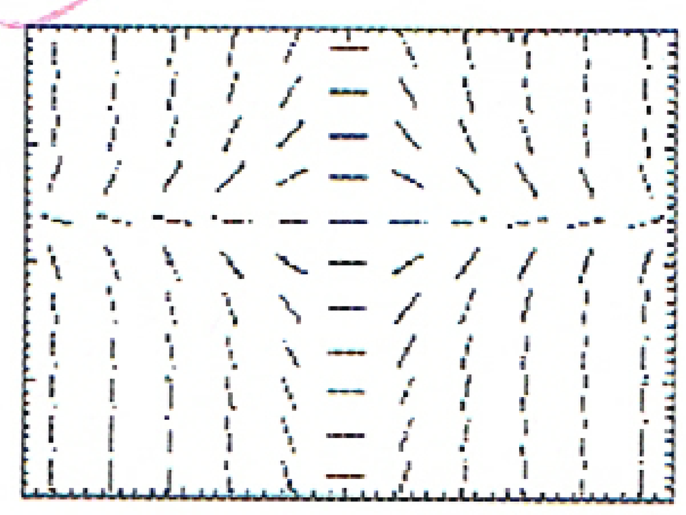
$$= e^{-t^3}\left[1 + \left(\frac{1}{3} e^u \Big|_{u=0}^{t^3}\right)\right]$$

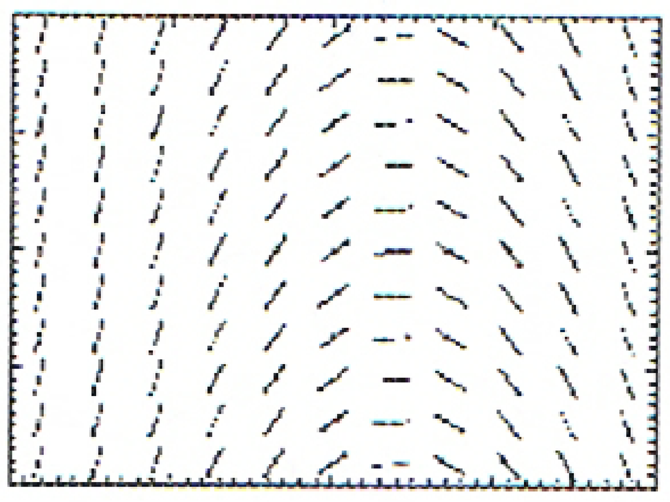
$$= e^{-t^3}\left[1 + \frac{e^{t^3}}{3} - \frac{1}{3}\right],$$

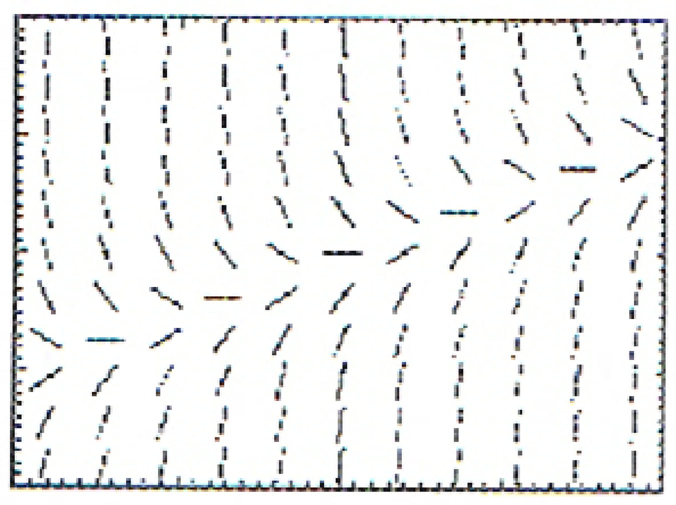
$$\boxed{x = e^{-t^3}\left[\frac{2}{3} + \frac{e^{t^3}}{3}\right]}$$

+4

(b) [2pts] Match the equations to their slope fields: (Fill in the blank with the letter)

a) 

b) 

c) 

[$y' = 1 - x$]: b.

Slope = 0 on $x = 1$

[$y' = x - 2y$]: c.

Slope = 0 on $x - 2y = 0$, $y = x/2$

[$y' = x(1-y)$]: a.

Slope = 0 on $y = 1$, and $x = 0$ lines.