Math 3D Quiz 2 Afternoon - April 20th
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Show all of your work. *Please go onto the back for more space for Problem 2*

1. [8 pts total] Let \( x' = 2x(3 - x) + 20 \). In other words, \( x' = -2x^2 + 6x + 20 \).
   (a) [4 pts] Find the equilibrium points (critical points) and sketch the phase diagram.

   **Critical Points:**
   \[
   \begin{align*}
   f(x) &= -2x^2 + 6x + 20 = 0, \\
   -2(x^2 - 3x - 10) &= 0, \\
   -2(x - 5)(x + 2) &= 0,
   \end{align*}
   \]
   \[x = -2, 5 \] +1

   **Phase Diagram:**
   \[
   \begin{array}{c}
   \text{Test Pts:} \\
   x = -3, f(-3) < 0 \\
   x = 0, f(0) > 0 \\
   x = 6, f(6) < 0
   \end{array}
   \]
   \[x = -2 \] +1
   \[x = 5 \] +1

   (b) [2 pts] Classify the critical points as stable or unstable.

   **Stable:** \( x = 5 \) +1
   **Unstable:** \( x = -2 \) +1

   (c) [2 pts] If \( x(0) = 0 \), what is \( \lim_{t \to \infty} x(t) \)? What is the limit if \( x(0) = 1000 \)?

   **Just to Recopy the phase diagram:**
   \[
   \begin{array}{c}
   \text{If } x(0) = 0: \quad \lim_{t \to \infty} x(t) = 5. \\
   \text{If } x(0) = 1000: \quad \lim_{t \to \infty} x(t) = 5.
   \end{array}
   \]

   This is 1.6103 with \( k = 2, m = 3, A = 20 \): Population of Fish
   Always limits to 5.
   (Because can't have negative fish)

2. [12 pts] Find the explicit solution to
   \[
y' + \frac{y}{x} + \frac{y^2}{x} = 0, \quad y(1) = 2.
   \]

   Even though this equation is separable, you must use a Bernoulli substitution for full credit.

   **Sol:** First rewrite as \( y' + \frac{y}{x} = -\frac{y^2}{x} \). Two ways to get the sub.

   **Way 1:** Profs' way, divide \( y^2 \),
   \[
   \frac{y'}{y^2} + \frac{1}{x} \cdot \frac{1}{y} = -\frac{1}{x^2}
   \]
   Set as \( V = \frac{1}{y} = \frac{1}{y} \).

   **Way 2:** From, identify \( n = 2 \),
   \[
   V = y^{1-n} = y^{-1} = \frac{1}{y} \]
   \[
   V' = -y^{-2}y'
   \]
so with way 1, \[ v' = -y' y \] 
so we can now substitute, 
\[ -v' + \frac{1}{x} v = -\frac{1}{x^2} \] 
\[ \text{Sub} \Rightarrow \ -v' + \frac{1}{x} v = -\frac{1}{x} \] 

Either way, we get 
\[ -v' + \frac{1}{x} v = -\frac{1}{x^2} \] 
\[ \Rightarrow \ y' y^{-2} + \frac{1}{x} y^{-1} = -\frac{1}{x} \] 
\[ \text{[ Need to multiply by } -1 \text{ for integrating factor] } \]

Now this can be solved in 2 ways, too.

**Way 1:** Integrating Factor,
\[ R(x) = e^{\int \frac{1}{x} \, dx} = e^{-\ln |x|} = \frac{1}{|x|} \]
we can drop abs. value because we multiply to both sides,
\[ \ln \left( \frac{1}{x} \cdot v \right) = \frac{1}{x^2} \]
Integrate, 
\[ \frac{1}{x} \cdot v = -\frac{1}{x} + C, \]
\[ v = cx - 1 \] 

**Way 2:** Fortunately separates,
\[ v' = \frac{1}{x} (v+1), \]
\[ \frac{dv}{v+1} = \frac{dx}{x}, \]
\[ \ln |v+1| = \ln |x| + C, \]
\[ |v+1| = c |x|, \]
Let \( C \) absorb signs,
\[ v+1 = cx, \quad v = cx-1 \] 

Either way, we get \( v = cx-1 \); \( y^{-1} = cx-1 \) after plugging back \( v = y^{-1} \).

I.C. for \( c \): 
\[ 2 = c - 1 \] ; \[ \frac{1}{2} = c - 1 \] ; \[ c = \frac{3}{2} \] 

So, \( y^{-1} = \frac{3}{2} x - 1 \) ; \( y = \frac{1}{\frac{3}{2} x - 1} \) 

Domain: \( x \neq \frac{2}{3} \) for denominator. \( x_0 = 1 \) is initially bigger than \( 2/3 \) so \( x > \frac{2}{3} \) is our domain.