Solutions

Math 3D Quiz 2 Afternoon - April 20th

Please put name on front & ID on back for redistribution!

Show all of your work. *Please go onto the back for more space for Problem 2*

1. [8 pts total] Let $x' = 2x(3-x) + 20$. In other words, $x' = -2x^2 + 6x + 20$.

(a) [4 pts] Find the equilibrium points (critical points) and sketch the phase diagram.

Critical Pts:

$f(x) = -2x^2 + 6x + 20 = 0$.

$-2(x^2 - 3x - 10) = 0$.

$-2(x-5)(x+2) = 0$,

$\boxed{x = -2, 5}$ +1

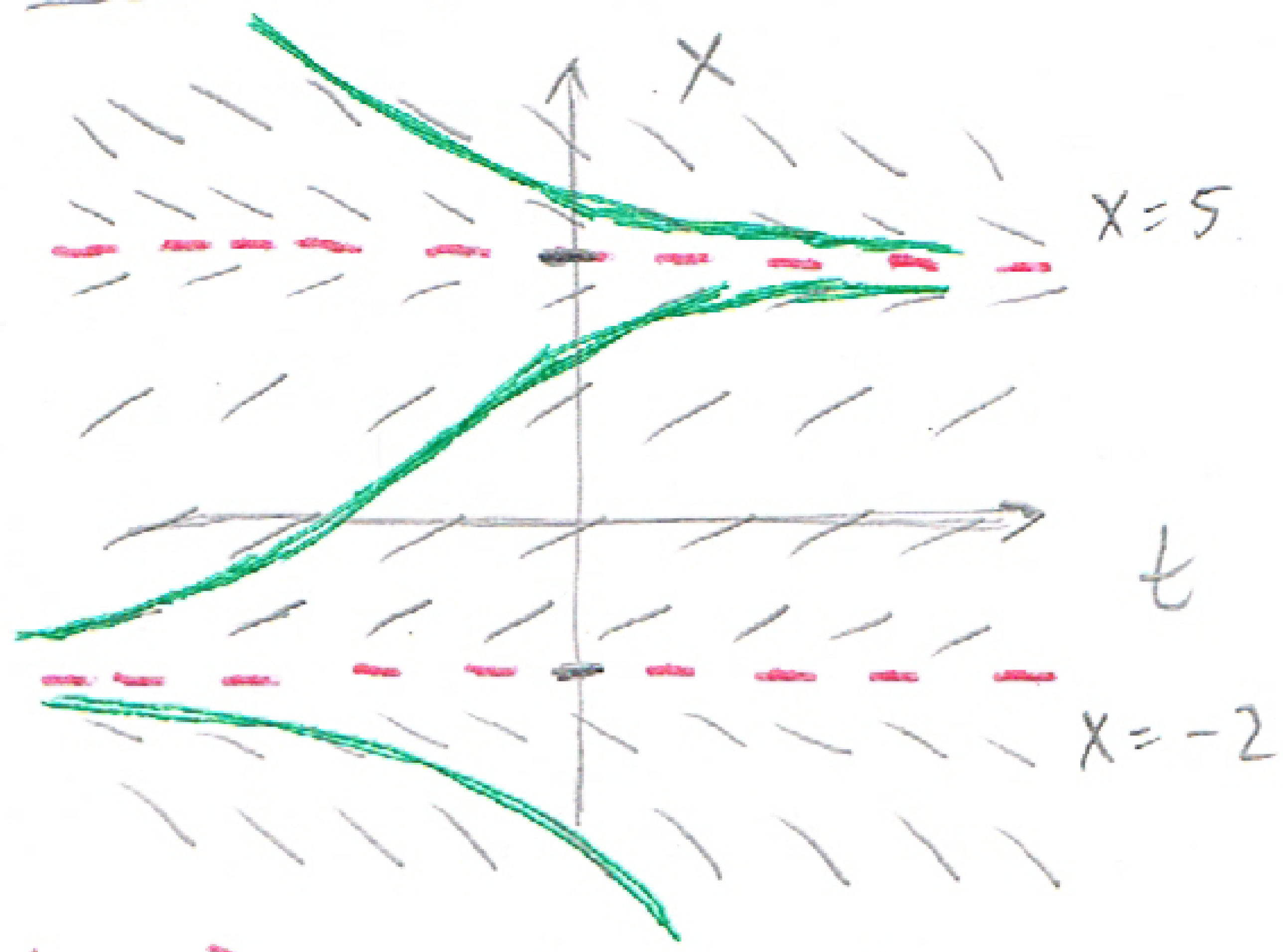
Phase Diagram:

+3

Test Pts:

- $x = -3$, $f(-3) < 0$
- $x = 0$, $f(0) > 0$
- $x = 6$, $f(6) < 0$

Not for Quiz, but for Studying: Slope Field,



(- -) ⇒ Steady Solns

(—) ⇒ sample solns.

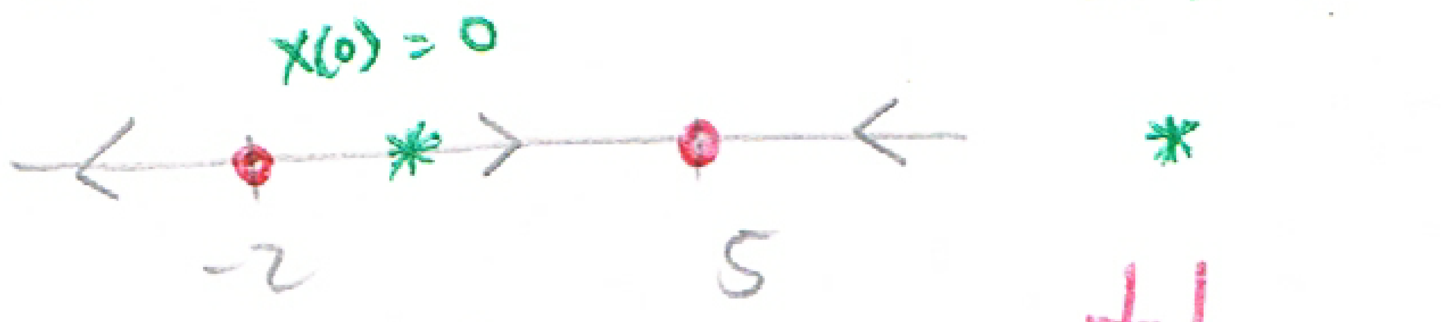
(b) [2 pts] Classify the critical points as stable or unstable.

Stable: $x = 5$ +1

unstable: $x = -2$ +1

(c) [2pts] If $x(0) = 0$, what is $\lim_{t\to\infty} x(t)$? What is the limit if $x(0) = 1000$?

Just to Recopy the phase diagram:



If $x(0) = 0$: $\lim_{t\to\infty} x(t) = 5$. +1 If $x(0) = 1000$: $\lim_{t\to\infty} x(t) = 5$. +1

This is 1.6.103 with $k=2, M=3, A=20$: Population of Fish. Always limits to 5. (Because can't have negative fish)

2. [12 pts] Find the explicit solution to

$$y' + \frac{y}{x} + \frac{y^2}{x} = 0, \qquad y(1) = 2.$$

Even though this equation is separable, you must use a Bernoulli substitution for full credit.

Sol: First Rewrite as $y' + \frac{y}{x} = -\frac{y^2}{x}$. Two ways to get the sub.

Way 1: Prof's way, divide y^2,

$$\frac{y'}{y^2} + \frac{1}{x}\cdot\boxed{\frac{1}{y}} = -\frac{1}{x^2}$$

set as $\boxed{v = \frac{1}{y} = y^{-1}}$

Way 2: From, identify $n = 2$,

So $\boxed{v = y^{1-n} = y^{-1} = \frac{1}{y}}$.

↳ $\boxed{v' = -y^{-2}y'}$

Soln Cont'd

So with $\underline{\text{Way 1}}$, $\boxed{v' = -y^{-2} y'}$,

so we can now substitute,

$$-v' + \frac{1}{x} \cdot v = -\frac{1}{x^2} \quad +3$$

With $\underline{\text{Way 2}}$, now we divide by y^2,

$$\Rightarrow y' y^{-2} + \frac{1}{x} y^{-1} = -\frac{1}{x},$$

$\underline{\text{sub}}$ $\Rightarrow$ $-v' + \frac{1}{x} v = -\frac{1}{x}$. $+3$

Either way, we get $-v' + \frac{1}{x} v = -\frac{1}{x^2}$ $\Rightarrow$ $\boxed{v' - \frac{1}{x} v = \frac{1}{x}}$ $+1$

[Need to multiply by -1 for integrating factor]

Now this can be solved in 2 ways, too.

$\underline{\text{Way 1}}$: Integrating Factor,

$$R(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{|x|}.$$

we can drop abs. value because we multiply to both sides,

$$\hookrightarrow \frac{d}{dx}\left(\frac{1}{x} \cdot v\right) = \frac{1}{x^2}.$$

Integrate, $\frac{1}{x} \cdot v = -\frac{1}{x} + c$,

$$\underline{v = cx - 1}. \quad +4$$

$\underline{\text{Way 2}}$: Fortunately separates,

$$v' = \frac{1}{x}(v+1),$$

$$\frac{dv}{v+1} = \frac{dx}{x},$$

$$\ln|v+1| = \ln|x| + \hat{c},$$

$$|v+1| = \tilde{c}\,|x|.$$

Let $\tilde{c}$ absorb signs,

$v + 1 = cx$, $\underline{v = cx - 1}$ $+4$

Either way, we get $v = cx - 1$; $\boxed{y^{-1} = cx - 1}$ after plugging back $v = y^{-1}$.

$\underline{\text{I.C. for } C}$: $2^{-1} = C - 1$; $\frac{1}{2} = C - 1$; $\boxed{C = \frac{3}{2}}$ $+2$

So, $y^{-1} = \frac{3}{2}x - 1$; $\boxed{y = \frac{1}{\frac{3x}{2} - 1}}$ $+1$

$\underline{\text{Domain}}$: $x \neq \frac{2}{3}$ for denominator. $x_0 = 1$ is initially bigger than $2/3$

$+1$ so $\boxed{x > \frac{2}{3}}$ is our domain.