Solutions

Math 3D Quiz 2 Morning - April 20th

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Show all of your work. *Please go onto the back for more space for Problem 2*

1. [8pts total] Consider the differential equation $y' = y(y-1)^2(y-2)$.

(a) [4pts] Find all the critical points (equilibrium solutions) and sketch the phase diagram.

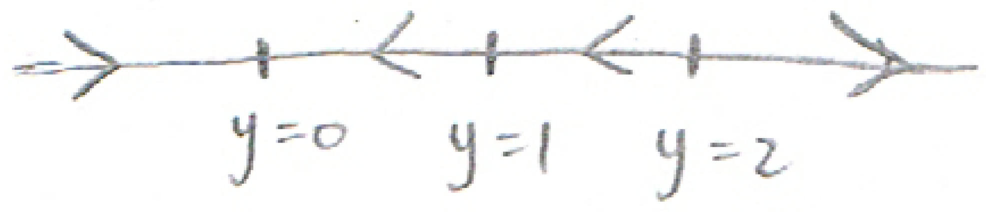
Crit. Pts:

$f(y) = y(y-1)^2(y-2) = 0$

$\hookrightarrow \boxed{y = 0, 1, 2}$ are crit. pts.

+1

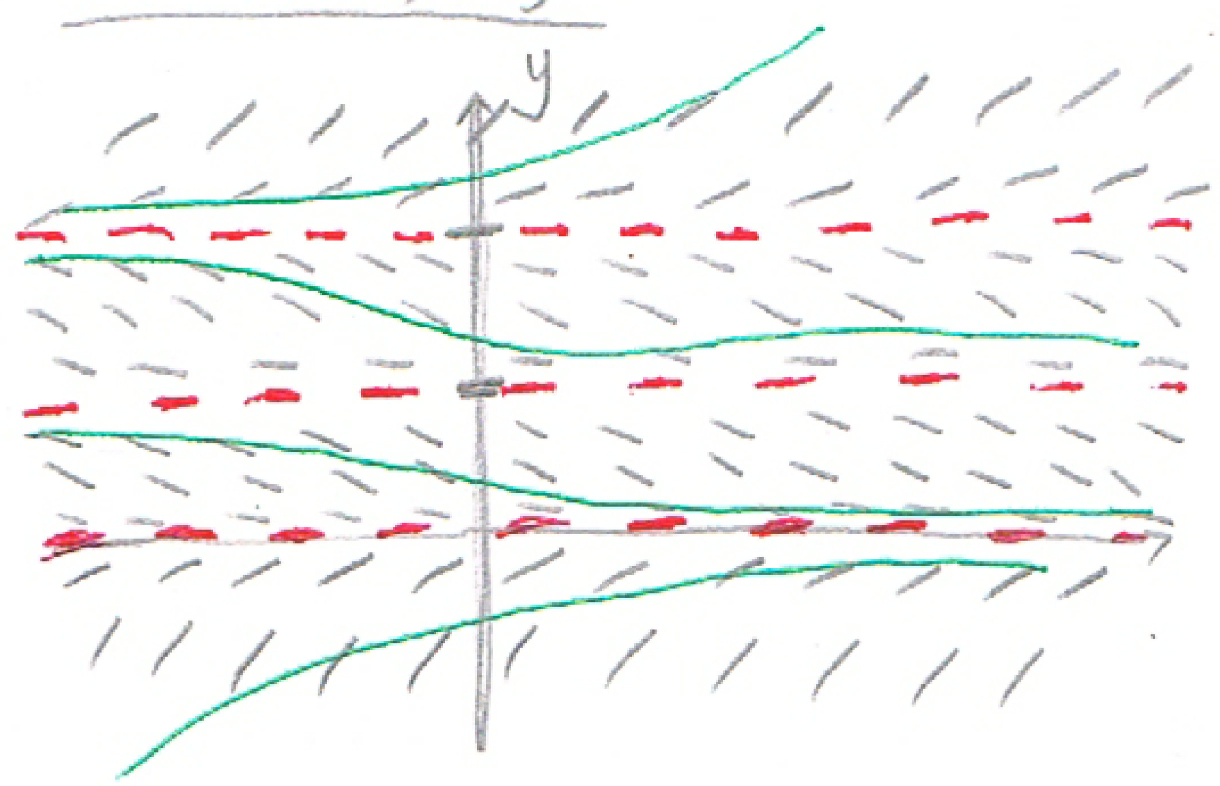
+3



Test Pts:

- $y=-1$, $f(-1) > 0$
- $y = \frac{1}{2}$, $f(\frac{1}{2}) < 0$
- $y = \frac{3}{2}$, $f(\frac{3}{2}) < 0$
- $y = 3$, $f(3) > 0$

Not for Quiz, but for studying:



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(- - -) ⇒ Steady solutions

(—) ⇒ Sample solutions.

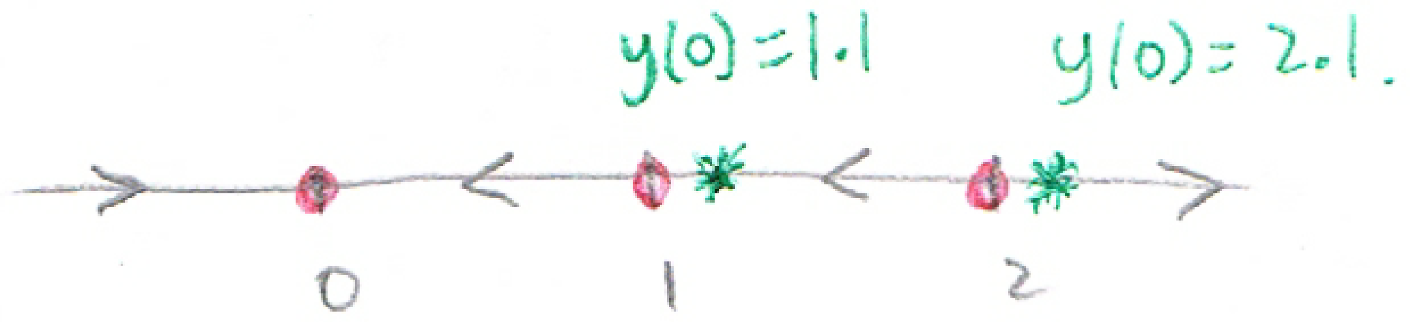
(b) [2pts] Classify the critical points as stable or unstable.

Stable: $y = 0$. +1

Unstable: $y = 1, 2$. +1

(c) [2pts] If $y(0) = 1.1$, what is $\lim_{t\to\infty} y(t)$? What is the limit if $y(0) = 2.1$ instead?

Just to recopy it:



From phase diagram: $y(0)=1.1$: $\lim_{t\to\infty} y(t) = 1$ +1

$y(0) = 2.1$: $\lim_{t\to\infty} y(t) = \infty$ +1

2. [12pts] Using a substitution, find the explicit solution to

$$xyy' = x^2 - y^2, \qquad y(1) = 2.$$

Hint: There are (at least) two viable substitutions. One is a homogeneous substitution. Another is one of the common substitutions on the table in the book. You can use either one.

Sol 1: Homogeneous (degree 2).

$\hookrightarrow$ First isolate, $y' = \frac{x^2 - y^2}{xy}$

so $y' = \frac{x}{y} - \frac{y}{x}$.

Sol 2: If we see yy' ⇝ Use $v = y^2$.

Sub $v = y^2$ ⇝ $v' = 2yy'$

so, equation becomes $\frac{xv'}{2} = x^2 - v$.

+2

Let $v = \frac{y}{x} \rightsquigarrow y = vx$ so $y' = xv' + v$,

$$xv' + v = \frac{1}{v} - v, \quad +3$$

$$xv' = \frac{1}{v} - 2v = \frac{1-2v^2}{v},$$

$$\frac{dv}{dx} = \left(\frac{1-2v^2}{v}\right) \cdot \frac{1}{x},$$

$$\int \frac{v}{1-2v^2}\,dv = \int \frac{dx}{x} \quad +1$$

let $u = 1-2v^2$, $du = -4v\,dv$,

$$\int -\frac{du}{4u} = \ln|x| + c,$$

$$-\frac{1}{4}\ln|u| = \ln|x| + c, \quad \text{(plug in } u\text{, mult. by } -4)$$

$\hookrightarrow \ln|1-2v^2| = -4\ln|x| + \tilde{c} \quad +2$

exp $\rightsquigarrow |1-2v^2| = \hat{c}\,|x|^{-4}$.

Absorb sign into $\hat{c}$, plug in v,

$+1 \quad 1 - 2\frac{y^2}{x^2} = \frac{\hat{c}}{x^4}$. [Can drop abs. value because 4th power]

I.C.: $1 - 2\cdot 4 = \hat{c}$, $\boxed{\hat{c} = -7}$ $+2$

$$1 - 2\frac{y^2}{x^2} = -\frac{7}{x^4}; \quad -\frac{2y^2}{x^2} = -1 - \frac{7}{x^4}$$

Solve for y: $\sqrt{y^2} = \pm\sqrt{\frac{1}{2}\left(x^2 + \frac{7}{x^2}\right)}$,

$+2 \quad \boxed{y = \sqrt{\frac{x^2}{2} + \frac{7}{2x^2}}}$, Domain: $\boxed{x > 0}$

Due to $+1$

Then, $\frac{v'}{2} = x - \frac{v}{x}$,

$+1 \quad v' + 2\frac{v}{x} = 2x$. [Linear Equ]

$$R(x) = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = |x|^2. \quad +2$$

$\hookrightarrow$ can drop absolute value b/c squared, & $R(x)$ multiplied to both sides

Then, $\frac{d}{dx}\left[x^2 \cdot v\right] = 2x^3$,

Integrate, $x^2 v = \frac{2x^4}{4} + c$,

$+2$

$$v = \frac{x^2}{2} + \frac{c}{x^2}.$$

Plug in $v = y^2$, $y^2 = \frac{x^2}{2} + \frac{c}{x^2}$.

I.C.: $4 = \frac{1}{2} + c$, $\boxed{c = \frac{7}{2}}$ $+2$,

$$y^2 = \frac{x^2}{2} + \frac{7}{2x^2},$$

$$y = \pm\sqrt{\frac{x^2}{2} + \frac{7}{2x^2}},$$

$+2 \quad \boxed{y = \sqrt{\frac{x^2}{2} + \frac{7}{2x^2}}}$.

[Chose $(+)$ branch because y is initially 2, 2 is positive]

Again, Domain is $\boxed{x > 0}$

$+1$

because initial $x = 1$ is positive.