Solns

Math 3D Quiz 4 Morning, May 4th

Please put name on front & ID on back for redistribution!

Show all of your work. *There is a question on the back side.*

1. (a) [3pts] Find the general solution to $y'' + 6y' + 8y = 0$.

Auxiliary/Characteristic Eqn: $r^2+6r+8=0$, $(r+2)(r+4)=0$, +1

Roots: $r=-2,-4$ +1 $\rightsquigarrow$ $\boxed{y_{gen} = c_1e^{-2x}+c_2e^{-4x}}$ +1

[This is y_c for (b) and (c)].

(b) [7pts] Find a particular solution to $y'' + 6y' + 8y = e^{-2x}$ using Variation of Parameters.

① Identify y_1, y_2 from (a): let $y_1 = e^{-2x}$, $y_2 = e^{-4x}$ +1

(order doesn't matter)

② Solve $\begin{cases} u_1'y_1+u_2'y_2=0 \\ u_1'y_1'+u_2'y_2'=f(x) \end{cases} \Rightarrow \begin{cases} u_1'e^{-2x}+u_2'e^{-4x}=0 \\ -2u_1'e^{-2x}-4u_2'e^{-4x}=e^{-2x} \end{cases}$ (already Standard form) +1

Use Either Approach:

Eliminate a variable: Eqn 2 + 2 · Eqn 1,

$\hookrightarrow -2u_2'e^{-4x}=e^{-2x}$, $\boxed{u_2'=-\frac{1}{2}e^{2x}}$

Plug back, $u_1'e^{-2x}-\frac{1}{2}e^{-2x}=0$, $\boxed{u_1'=\frac{1}{2}}$

So $\boxed{u_1'=\frac{1}{2},\ u_2'=-\frac{1}{2}e^{2x}}$ +2

Cramer's Rule:

$W = \det\begin{vmatrix} e^{-2x} & e^{-4x} \\ -2e^{-2x} & -4e^{-4x} \end{vmatrix} = -2e^{-6x}$

$W_1 = \det\begin{vmatrix} 0 & e^{-4x} \\ e^{-2x} & -4e^{-4x} \end{vmatrix} = -e^{-6x}$

$W_2 = \det\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-2x} \end{vmatrix} = e^{-4x}$.

$\boxed{u_1' = \frac{W_1}{W} = \frac{1}{2},\quad u_2' = \frac{W_2}{W} = -\frac{1}{2}e^{2x}}$

Either way, $u_1' = \frac{1}{2} \Rightarrow \boxed{u_1 = \frac{x}{2}}$ +1, $u_2' = -\frac{e^{2x}}{2} \Rightarrow \boxed{u_2 = -\frac{e^{2x}}{4}}$ +1

So $y_p = u_1y_1+u_2y_2 \Rightarrow \boxed{y_p = \frac{x}{2}e^{-2x}-\frac{e^{-2x}}{4}}$ +1

(c) [7pts] Now, find a particular solution to $y'' + 6y' + 8y = e^{-2x}$ using Undetermined Coefficients.

* For RHS $= e^{-2x}$, would guess $y_p = Ae^{-2x}$

* Since from (a), $y_c = \underline{C_1 e^{-2x}} + C_2 e^{-4x}$, need to scale by x

↳ So our guess is actually $\boxed{y_p = Axe^{-2x}}$ +3

Then $y_p' = A(e^{-2x} - 2xe^{-2x})$ +1 ; $y_p'' = A(-2e^{-2x} - 2e^{-2x} + 4xe^{-2x})$

$= A(-4e^{-2x} + 4xe^{-2x})$ +1

Plug in,

$$A(-4e^{-4x} + 4xe^{-4x}) + 6A(e^{-2x} - 2xe^{-2x}) + 8Axe^{-2x} = e^{-2x}$$

Function Types & Eqns:

xe^{-4x} : $4A - 12A + 8A = 0$ ✓ (consistent)

e^{-4x} : $-4A + 6A = 1$, $\boxed{A = \frac{1}{2}}$ +1

Thus, $A = \frac{1}{2} \Rightarrow \boxed{y_p = \frac{x}{2} e^{-2x}}$ +1

(d) [1pt] Are your answers in (b) and (c) the same? If not, are both answers valid? Explain.

+1 Not the same, but both valid: (b) has an extra $\left(-\frac{e^{-2x}}{4}\right)$ term, a part of y_c, so it goes to 0 when plugged into the equation

(e) [2pts] Reduce the order of the system in (b),(c) into a 1st order matrix vector equation.

2^{nd} order $\Rightarrow$ 2 variables (2 equations)

$$\left.\begin{array}{l} u_0 = y \\ u_1 = y' \\ y'' = -6y' - 8y + e^{-2x} \end{array}\right\} \begin{array}{l} u_0' = u_1 \quad +1 \\ u_1' = -6u_1 - 8u_0 + e^{-2x} \end{array} \Rightarrow \boxed{\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-2x} \end{bmatrix}}$$

+1

in other words, $\boxed{\begin{bmatrix} y \\ y' \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ e^{-2x} \end{bmatrix}}$