Solutions

Math 3D Quiz 5 Morning - May 11th

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Show all of your work. *There is a question on the back side.*

1. Consider the first order system $x_1' = 5x_2$, $x_2' = -5x_1$, $x_3' = 7x_3$.

(a) [10pts] Find the general solution using the Eigenvalue Method.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 5 & 0 \\ -5 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

① Eigenvalues: $\det \begin{bmatrix} -\lambda & 5 & 0 \\ -5 & -\lambda & 0 \\ 0 & 0 & 7-\lambda \end{bmatrix} = (7-\lambda)\cdot\det\begin{bmatrix} -\lambda & 5 \\ -5 & -\lambda \end{bmatrix} = (7-\lambda)(\lambda^2+25)$

So $\boxed{\lambda = 7, \pm 5i}$ +2

② Eigenvectors: $\lambda = 7$: $\left[\begin{array}{ccc|c} -7 & 5 & 0 & 0 \\ -5 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_2 - \frac{5}{7}R_1}{\sim} \left[\begin{array}{ccc|c} -7 & 5 & 0 & 0 \\ 0 & -7-\frac{25}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

says $v_2 = 0$, so $v_1 = 0$ and $v_3 =$ free, $\boxed{\vec{v} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$ +2

$\lambda = 5i$: $\left[\begin{array}{ccc|c} -5i & 5 & 0 & 0 \\ -5 & -5i & 0 & 0 \\ 0 & 0 & 7-5i & 0 \end{array}\right] \Rightarrow$ $-5iv_1 + 5v_2 = 0$, so $v_2 = iv_1$, $v_1 =$ free, $v_3 = 0$

$\boxed{\vec{v} = v_1 \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}}$ +2

Yields complex soln $e^{i5t}\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = (\cos 5t + i\sin 5t)\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

Real: $\begin{bmatrix} \cos 5t \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \sin 5t \\ 0 \end{bmatrix}$ +1

Imag: $(i)\begin{bmatrix} 0 \\ \cos 5t \\ 0 \end{bmatrix} + (i)\begin{bmatrix} \sin 5t \\ 0 \\ 0 \end{bmatrix}$ +1

The (i) will be dropped in

So, $$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 e^{7t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + C_2\begin{bmatrix} \cos 5t \\ -\sin 5t \\ 0 \end{bmatrix} + C_3\begin{bmatrix} \sin 5t \\ \cos 5t \\ 0 \end{bmatrix}}$$

+2

$\boxed{\text{Solns Cont'd}}$

(b) [4pts] Prove that the vector functions from the general solution in (a) are Linearly Independent.
Hint: Just plug in $t = 0$.

The functions are $\left\{ e^{7t}\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}\cos 5t\\-\sin 5t\\0\end{bmatrix}, \begin{bmatrix}\sin 5t\\\cos 5t\\0\end{bmatrix} \right\}$.

Need $c_1 e^{7t}\begin{bmatrix}0\\0\\1\end{bmatrix} + c_2\begin{bmatrix}\cos 5t\\-\sin 5t\\0\end{bmatrix} + c_3\begin{bmatrix}\sin 5t\\\cos 5t\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

+3

At $t=0$: $c_1\begin{bmatrix}0\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\\0\end{bmatrix} + c_3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

As matrix, or just componentwise $\Rightarrow$ says c_1, c_2, c_3 are all zero

$\Rightarrow$ set is L.I. ✓ +1

(c) [6pts] Now solve $x_1' = 5x_2$, $x_2' = -5x_1$, $x_3' = 7x_3$ by solving for each component.
Here is a Hint / Outline:
First solve for x_3 on its own. Then decouple the x_1', x_2' equations to solve for those components.

* Solve x_3: $x_3' = 7x_3$, $x_3' - 7x_3 = 0$, Like $0x_3'' + x_3' - 7x_3 = 0$

Aux Equ: $r - 7 = 0$, $r = 7$, $\boxed{x_3 = c_3 e^{7t}}$ +2

* Decouple x_1, x_2: Take further derivative, like $x_1'' = 5x_2'$
$= -25x_1$

So $x_1'' = -25x_1$, $x_1'' + 25x_1 = 0$

+3

Aux Equ: $r^2 + 25 = 0$, $r = \pm 5i$, $\boxed{x_1 = c_1\cos 5t + c_2 \sin 5t}$

Then $x_2 = \dfrac{x_1'}{5}$ $\Rightarrow$ $x_2 = \frac{1}{5}\left(-5c_1\sin 5t + 5c_2\cos 5t\right)$

+1

$\boxed{x_2 = -c_1\sin 5t + c_2\cos 5t}$

☆ Note: These are the same as the components in (a), but with renamed constants.