$\boxed{\text{Solution}}$

Math 3D Quiz 6 Afternoon - May 18th

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Show all of your work. *There is a question on the back side.*

1. Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. (a) What are the eigenvalues and eigenvectors?

(b) What are the defects? (c) Find the general solution to $\vec{x}\,' = A\vec{x}$.

(a) Note A ~ upper triangular, eigenvalues are on main diagonal, $\boxed{\lambda = 2 \ (AM = 4)}$ +1

Check: $\det \begin{bmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = (2-\lambda)\det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$

$= (2-\lambda)^2 \cdot \det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^4,$

(GM = 2)

Eigenvectors, Solve $(A-2I)\vec{v} = \vec{0}$: $\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $v_1 = \text{free}$, $v_2 = 0$, $v_3 = \text{free}$, $v_4 = 0$, $\boxed{\vec{v} = v_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + v_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}$

(Two Eigenvectors) +2

(b) $\lambda = 2$ is twice defective +1 because $AM - GM = 4 - 2 = 2$.

(c) Need two more generalized eigenvectors. Note: Both work!

$(A-2I)\vec{u} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$:

$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $u_1 = \text{free}$, $u_2 = 1$, $u_3 = \text{free}$, $u_4 = 0$.

$\boxed{\vec{u} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + u_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + u_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}$

(1st Generalized eigenvector from) +2

$(A-2I)\vec{w} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$,

$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$ $w_1 = \text{free}$, $w_2 = 0$, $w_3 = \text{free}$, $w_4 = 1$,

$\boxed{\vec{w} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} + w_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + w_3\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}$ +2

(1st Generalized eigenvector from)

So the general solution is

$$\boxed{\vec{x}_{general} = c_1 e^{2t}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + c_2 e^{2t}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right) + c_3 e^{2t}\begin{bmatrix}0\\0\\1\\0\end{bmatrix} + c_4 e^{2t}\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix} + t\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right)}$$ +2

Eigenvector Decomposition:

Is $\vec{f}(t)$

2. Find the general solution to $\vec{x}\,' = \begin{bmatrix} 5 & 4 \\ 1 & 8 \end{bmatrix} \vec{x} + \boxed{\begin{bmatrix} t \\ -t \end{bmatrix}}$.

$\lambda = 4, \quad \vec{v}_4 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$\lambda = 9, \quad \vec{v}_9 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hints: Use that the complementary solution is $\vec{x}_c = C_1 e^{4t} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + C_2 e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If you prefer Eigenvector Decomposition: Read the eigenvalues and eigenvectors from $\vec{x}_c$.
If you prefer Undetermined Coefficients: "All" you need to do is solve for $\vec{x}_p$ now.

First construct E, like $E = \begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$ and $E^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix}$.

+1

$\uparrow \vec{v}_4 \quad \uparrow \vec{v}_9$

* Get $g_1(t), g_2(t)$:

Way 1: $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1} \vec{f}$,

$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} t \\ -t \end{bmatrix}$

$= \frac{1}{5} \begin{bmatrix} 2t \\ -3t \end{bmatrix}$

Way 2: $E \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \vec{f}$,

$\left[\begin{array}{cc|c} 4 & 1 & t \\ -1 & 1 & -t \end{array}\right] \overset{R_1 + 4R_2}{\sim} \left[\begin{array}{cc|c} 0 & 5 & -3t \\ -1 & 1 & -t \end{array}\right]$

Row 1: $5 g_2 = -3t$, $\boxed{g_2 = -\frac{3t}{5}}$,

Row 2: $-g_1 + g_2 = -t$, $g_1 = g_2 + t$, $\boxed{g_1 = \frac{2t}{5}}$

Either way $g_1 = 2t/5$, $g_2 = -3t/5$. +2

* Set up/solve our eqns for $\varepsilon_1, \varepsilon_2$. Note: $\lambda = 4 \sim 1^{st}$, $\lambda = 9 \; 2^{nd}$!!

1) $\varepsilon_1' = 4\varepsilon_1 + 2t/5$,

+1 $\varepsilon_1' - 4\varepsilon_1 = 2t/5$

☆ Need ε_{1c}, ε_{1p}.

For ε_{1c}: Aux Eqn $r - 4 = 0$, $r = 4$,

$\boxed{\varepsilon_{1c} = c_1 e^{4t}}$

For ε_{1p}: Guess $\varepsilon_{1p} = At + B$, $\varepsilon_{1p}' = A$.

Plug in $\rightsquigarrow$ $A - 4(At + B) = 2t/5$,

Fn Types: t: $-4A = 2/5$, $\boxed{A = -\frac{1}{10}}$

1: $A - 4B = 0$, $\boxed{B = \frac{A}{4} = -\frac{1}{40}}$

So $\boxed{\varepsilon_1 = \varepsilon_{1c} + \varepsilon_{1p} = c_1 e^{4t} - \frac{t}{10} - \frac{1}{40}}$ +2

2) $\varepsilon_2' = 9\varepsilon_2 - 3t/5$,

+1 $\varepsilon_2' - 9\varepsilon_2 = -3t/5$

☆ Need $\varepsilon_{2c}, \varepsilon_{2p}$

For ε_{2c}: Aux Eqn $r - 9 = 0$, $r = 9$,

$\boxed{\varepsilon_{2c} = C_2 e^{9t}}$

For ε_{2p}: Guess $\varepsilon_{2p} = At + B$, $\varepsilon_{2p}' = A$

Plugin, $A - 9(At + B) = -3t/5$,

Fn Types: t: $-9A = -3/5$, $\boxed{A = \frac{1}{15}}$

1: $A - 9B = 0$, $B = \frac{A}{9}$, $\boxed{B = \frac{1}{135}}$

So $\boxed{\varepsilon_2 = \varepsilon_{2c} + \varepsilon_{2p} = C_2 e^{9t} + \frac{t}{15} + \frac{1}{135}}$ +

* Plug into $\vec{X} = \varepsilon_1 \vec{v}_1 + \varepsilon_2 \vec{v}_2$,

$$\boxed{\vec{X} = \left(c_1 e^{4t} - \frac{t}{10} - \frac{1}{40}\right) \begin{bmatrix} 4 \\ -1 \end{bmatrix} + \left(c_2 e^{9t} + \frac{t}{15} + \frac{1}{135}\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$ +1

Undetermined Coefficients

2. Find the general solution to $\vec{x}\,' = \begin{bmatrix} 5 & 4 \\ 1 & 8 \end{bmatrix} \vec{x} + \boxed{\begin{bmatrix} t \\ -t \end{bmatrix}}$. Is $\vec{f}(t)$.

Hints: Use that the complementary solution is $\vec{x}_c = C_1 e^{4t} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + C_2 e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

If you prefer Eigenvector Decomposition: Read the eigenvalues and eigenvectors from $\vec{x}_c$.
If you prefer Undetermined Coefficients: "All" you need to do is solve for $\vec{x}_p$ now.

For $\vec{X}_p$, Guess: $\vec{X}_p = t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ based on $\vec{f}(t)$.

And, then $\vec{X}_p{}' = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. +2

(Note $\vec{X}_p$ is not part of $\vec{X}_c$ ✓)
No shift needed

Plugin, $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 1 & 8 \end{bmatrix} \left(t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + \begin{bmatrix} t \\ -t \end{bmatrix}$,

+1

$$= \begin{bmatrix} t(5a_1+4a_2) + (5b_1+4b_2) \\ t(a_1+8a_2) + (b_1+8b_2) \end{bmatrix} + \begin{bmatrix} t \\ -t \end{bmatrix}.$$

Row 1: $a_1 = t(5a_1+4a_2+1) + (5b_1+4b_2)$

Fn Types: t: $0 = 5a_1+4a_2+1$

+1 1: $a_1 = 5b_1+4b_2$

Row 2: $a_2 = t(a_1+8a_2-1) + (b_1+8b_2)$

Fn Types: t: $0 = a_1+8a_2-1$

+1 1: $a_2 = b_1+8b_2$

* Use the "t" eqns to get a_1, a_2: $\left.\begin{matrix} 5a_1+4a_2 = -1 \\ a_1+8a_2 = +1 \end{matrix}\right\}$ $9a_1 = -3$, $\boxed{a_1 = -\frac{1}{3}}$

So $\boxed{a_2 = \frac{1}{6}}$ +2

* Then, $\left.\begin{matrix} 5b_1+4b_2 = -1/3 \\ b_1+8b_2 = 1/6 \end{matrix}\right\}$ $9b_1 = -5/6$, $\boxed{b_1 = -\frac{5}{54}}$ +2

So $8b_2 = \frac{1}{6} + \frac{5}{54} = \frac{14}{54}$, $\boxed{b_2 = \frac{7}{4 \cdot 54} = \frac{7}{216}}$

So $\vec{X}_p = t \begin{bmatrix} -1/3 \\ 1/6 \end{bmatrix} + \begin{bmatrix} -5/54 \\ 7/216 \end{bmatrix}$,

$$\boxed{\vec{X}_{general} = C_1 e^{4t} \begin{bmatrix} 4 \\ -1 \end{bmatrix} + C_2 e^{9t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1/3 \\ 1/6 \end{bmatrix} + \begin{bmatrix} -5/54 \\ 7/216 \end{bmatrix}}$$

+1

Check w/ Eigendecomposition:

The particular part is $\left(-\frac{t}{10} - \frac{1}{40}\right)\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \left(\frac{t}{15} + \frac{1}{135}\right)\Big[$

Row 1: $-\frac{4t}{10} - \frac{4}{40} + \frac{t}{15} + \frac{1}{135}$

(Do same with Row 2) $= -\frac{2t}{5} - \frac{1}{10} + \frac{t}{15} + 1/135$

$= -\frac{5}{15}t - \frac{1}{10} + 1/135 \longrightarrow -\frac{t}{3} - \frac{5}{54}$