Solutions

Math 3D Quiz 6 Morning - May 18th

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Show all of your work. *There is a question on the back side.*

1. Consider the matrix $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$. (a) What are the eigenvalues and eigenvectors? (b) What are the defects? (c) Find the general solution to $\vec{x}\,' = A\vec{x}$.

(a) Note A is upper triangular, eigenvalues are on main diagonal, $\boxed{\lambda = 3 \ (\text{AM } 4)}$ +1

Check: $\det(A-\lambda I) = \det \begin{bmatrix} 3-\lambda & 1 & 0 & 0 \\ 0 & 3-\lambda & 1 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} = (3-\lambda)\det \begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}$

$= (3-\lambda)^2 \cdot \det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 \cdot (3-\lambda)^2,$

Eigenvectors: $(A-3I)\vec{v} = \vec{0}$

$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} v_1 = \text{free} \\ v_2 = 0 \\ v_3 = 0 \\ v_4 = \text{free} \end{array}$, $\boxed{\vec{V} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$ +2

(Two eigenvectors)

(b) $\lambda = 3$ is defective twice, +1 AM − GM = 4 − 2 = 2. (GM = 2)

(c) Need generalized eigenvectors. Note: That $(A-3I)\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ has NO solution!!

↳ Solve: $(A-3I)\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$: $\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} u_1 = \text{free} \\ u_2 = 1 \\ u_3 = 0 \\ u_4 = \text{free} \end{array}$, $\boxed{\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$ +2

(1st Generalized Eigenvector)

Need one more, solve: $(A-3I)\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} w_1 = \text{free} \\ w_2 = 0 \\ w_3 = 1 \\ w_4 = \text{free} \end{array}$, $\boxed{\vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + w_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$ +2

(2nd Generalized Eigenvector)

So the general solution is

$$\boxed{\vec{X}_{\text{general}} = c_1 e^{3t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + c_4 e^{3t} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)}$$

+2

Eigenvector Decomposition

2. Find the general solution to $\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} + \boxed{\begin{bmatrix} t \\ t \end{bmatrix}}$. Is $\vec{f}(t)$.

$\rightarrow$ $\lambda = 1$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Hints: Use that the complementary solution is $\vec{x}_c = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\lambda = -1$, $\vec{v}_{-1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

If you prefer Eigenvector Decomposition: Read the eigenvalues and eigenvectors from $\vec{x}_c$.
If you prefer Undetermined Coefficients: "All" you need to do is solve for $\vec{x}_p$ now.

+1 First we need to construct E, like $E = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ so $E^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

* Get $g_1(t), g_2(t)$:

Way 1: $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1}\vec{f}$

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} t \\ t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2t \\ 0 \end{bmatrix}.$$

Way 2: Solve $E\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \vec{f}$,

$$\left[\begin{array}{cc|c} 1 & -1 & t \\ 1 & 1 & t \end{array}\right] \overset{R_2 - R_1}{\sim} \left[\begin{array}{cc|c} 1 & -1 & t \\ 0 & -2 & 0 \end{array}\right],$$

Row 2: $g_2 = 0 \Rightarrow g_1 = t$.

+2 for g's.

* Setup/solve our eqns:

$\lambda = 1$ 1st!
$\lambda = -1$ 2nd!

1) $\varepsilon_1' = \varepsilon_1 + g_1(t)$ +1

$\hookrightarrow$ $\varepsilon_1' - \varepsilon_1 = t$ // Need $\varepsilon_{1p}, \varepsilon_{1c}$.

For ε_{1c}: Aux Eqn $r - 1 = 0$, $r = 1$,

$\varepsilon_{1c} = c_1 e^t$.

For ε_{1p}: Guess $\varepsilon_{1p} = At + B$, $\varepsilon_{1p}' = A$,

Plugin, $A - (At + B) = t$,

Fn Types: t: $-A = 1$, $\boxed{A = -1}$

1: $A - B = 0$, $\boxed{B = A = -1}$

$\varepsilon_{1p} = -t - 1$ so

$\boxed{\varepsilon_1 = c_1 e^t - t - 1}$ +2

$(\varepsilon_{1c} + \varepsilon_{1p})$

2) $\varepsilon_2' = -\varepsilon_2 + g_2(t)$. +1

$\hookrightarrow$ $\varepsilon_2' + \varepsilon_2 = 0$.

Aux. Eqn: $r + 1 = 0$, $r = -1$,

so $\boxed{\varepsilon_2 = c_2 e^{-t}}$ +2

* Plug into solution: $\vec{x} = \varepsilon_1(t)\vec{v}_1 + \varepsilon_2(t)\vec{v}_2 = \varepsilon_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \varepsilon_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$,

+1

$$\boxed{\vec{x} = (c_1 e^t - t - 1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

is our general solution

Undetermined Coefs (for 3.9, not 2.5)

2. Find the general solution to $\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} + \boxed{\begin{bmatrix} t \\ t \end{bmatrix}}$. = Is $\vec{f}(t)$!

Hints: Use that the complementary solution is $\vec{x}_c = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

If you prefer Eigenvector Decomposition: Read the eigenvalues and eigenvectors from $\vec{x}_c$.
If you prefer Undetermined Coefficients: "All" you need to do is solve for $\vec{x}_p$ now.

For $\vec{x}_p$, Guess: $\vec{x}_p = t\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ based on.

~~+3~~ +2

- Note $\vec{x}_p$ is not part of $\vec{x}_c$ so no shift needed ✓

$$\vec{x}_p = t\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow \vec{x}_p\,' = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Plugin, $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right) + \begin{bmatrix} t \\ t \end{bmatrix}$,

+1

$$= \begin{bmatrix} a_2 t + b_2 \\ a_1 t + b_1 \end{bmatrix} + \begin{bmatrix} t \\ t \end{bmatrix}.$$

Row 1: $a_1 = a_2 t + b_2 + t$

Fn Types: $t: \; 0 = a_2 + 1$
$1: \; a_1 = b_2$

+1

So $\boxed{a_2 = -1}$ +1 and also

so $\boxed{b_1 = a_2 = -1 \text{ too}}$ +1

Row 2: $a_2 = a_1 t + b_1 + t$.

Fn Types: $t: \; 0 = a_1 + 1$
$1: \; a_2 = b_1$

+1

$\boxed{a_1 = -1}$ +1

So $\boxed{b_2 = a_1 = -1}$ too +1

Thus $\boxed{\vec{x}_p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}t + \begin{bmatrix} -1 \\ -1 \end{bmatrix}},$

So
$$\boxed{\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}}.$$

+1

(as $\vec{x}_c + \vec{x}_p$)

Check: From Eigendecomposition the particular piece is only due to $(-t-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and this matches here. ✓