Solutions

Math 3D Quiz 7 Afternoon - May 25th

Please put name on front & ID on back for redistribution!

Show all of your work. *There is a question on the back side.*

1. (a) [2pts] Fill in the blank. What are the following shifting properties?
(You can let $F(s)$ denote $\mathcal{L}\{f(t)\}$, the Laplace Transform of $f(t)$, if you'd like.)

$$\mathcal{L}\{e^{-at}f(t)\} = \underline{F(s+a)} \quad +1$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = \underline{e^{-as}F(s)} \quad +1$$

(b) [8pts] Using the Laplace Transform, solve $x'' + 2x' + 10x = 0$ with $x(0) = 1$, $x'(0) = 3$.

Transform the Eqn: $(s^2 X(s) - s - 3) + 2(sX(s) - 1) + 10X(s) = 0$. +1

Isolate $X(s)$: $X(s)\cdot(s^2+2s+10) = s+3+2$; $\boxed{X(s) = \dfrac{s+5}{s^2+2s+10}}$ +1

Note: $s^2+2s+10$ has complex roots, $s = \dfrac{-2\pm\sqrt{4-40}}{2}$, $s = \dfrac{-2\pm 6i}{2}$,

↳ complete the square! And impose its shift. (of $s+1$)

$$X(s) = \frac{s+5}{(s+1)^2+9} = \frac{(s+1)+4}{(s+1)^2+9}$$

+3

so, $X(s) = \dfrac{s+1}{(s+1)^2+9} + \dfrac{4}{(s+1)^2+9}$ ☆ Now we can start inverse transforming.

$$x(t) = \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+9}\right) + \mathcal{L}^{-1}\left(\frac{4}{(s+1)^2+9}\right)$$

shifting prop

$$= e^{-t}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + 4e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right)\cdot\frac{3}{3}$$

so,

$$\boxed{x(t) = e^{-t}\cos(3t) + \frac{4}{3}e^{-t}\sin(3t)}$$ +3

Undetermined Coefficients

$= \vec{f}(t) = e^t \begin{bmatrix}1\\0\end{bmatrix}.$

2. [10pts] Find the general solution to $\vec{x}\,' = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\vec{x} + \boxed{\begin{bmatrix}e^t\\0\end{bmatrix}}$.

Hints: Use that the complementary solution is $\vec{x}_c = C_1 e^t \begin{bmatrix}1\\1\end{bmatrix} + C_2 e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$.

If you prefer Undetermined Coefficients: "All" you need to do is solve for $\vec{x}_p$ now, and add it to $\vec{x}_c$.

If you prefer Eigenvector Decomposition: Read the eigenvalues and eigenvectors from $\vec{x}_c$.

1) Have $\vec{x}_c$ already ✓

2) for $\vec{f}(t) = e^t\begin{bmatrix}1\\0\end{bmatrix}$, overlaps with $\vec{x}_c$ once, so

Guess: $\vec{x}_p = e^t\begin{bmatrix}a_1\\a_2\end{bmatrix} + te^t\begin{bmatrix}b_1\\b_2\end{bmatrix}$. +2

Need to plug into our eqn, $\vec{x}_p\,' = \begin{bmatrix}0&1\\1&0\end{bmatrix}\vec{x}_p + \begin{bmatrix}e^t\\0\end{bmatrix}$,

$\vec{x}_p\,' = e^t\begin{bmatrix}a_1\\a_2\end{bmatrix} + (e^t + te^t)\begin{bmatrix}b_1\\b_2\end{bmatrix} = e^t\begin{bmatrix}a_1+b_1\\a_2+b_2\end{bmatrix} + te^t\begin{bmatrix}b_1\\b_2\end{bmatrix}$.

$\begin{bmatrix}0&1\\1&0\end{bmatrix}\vec{x}_p = \begin{bmatrix}0&1\\1&0\end{bmatrix}\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}e^t + \begin{bmatrix}b_1\\b_2\end{bmatrix}te^t\right)$

$= e^t\begin{bmatrix}a_2\\a_1\end{bmatrix} + te^t\begin{bmatrix}b_2\\b_1\end{bmatrix}$.

So, $e^t\begin{bmatrix}a_1+b_1\\a_2+b_2\end{bmatrix} + te^t\begin{bmatrix}b_1\\b_2\end{bmatrix} = te^t\begin{bmatrix}b_2\\b_1\end{bmatrix} + e^t\begin{bmatrix}a_2+1\\a_1\end{bmatrix}$. +1

Row 1: $e^t(a_1+b_1) + te^t(b_1) = te^t(b_2) + e^t(a_2+1)$.

Row 2: $e^t(a_2+b_2) + te^t(b_2) = te^t(b_1) + e^t(a_1)$. +2

Row 1, e^t: $a_1+b_1 = a_2+1 \Rightarrow a_1 - a_2 + b_1 = 1$

Row 1, te^t: $b_1 = b_2$ ✓

(since $b_2 = b_1$)

Row 2, e^t: $a_2 + b_2 = a_1 \Rightarrow -a_1 + a_2 + b_1 = 0$.

Row 2, te^t: $b_2 = b_1$ ✓

Add these, $2b_1 = 1$, $\boxed{b_1 = \frac{1}{2}}$ so $\boxed{b_2 = \frac{1}{2}}$ too. +2

Then we have $a_1 - a_2 + \frac{1}{2} = 1 \Rightarrow a_1 - a_2 = \frac{1}{2}$

$-a_1 + a_2 + \frac{1}{2} = 0 \Rightarrow -a_1 + a_2 = -1/2$

Eqns are same $\Leftrightarrow$ free var!

So, since same eqn, let $a_1 = a_2 + 1/2$, $a_2 =$ free +2 $\Rightarrow$ set $\boxed{a_2 = 0}$ so $\boxed{a_1 = 1/2}$

and plug coefs into $\vec{x}_p$.

So, $\boxed{\vec{X} = \vec{X}_c + \vec{X}_p = c_1 e^t\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + e^t\begin{bmatrix}1/2\\0\end{bmatrix} + te^t\begin{bmatrix}1/2\\1/2\end{bmatrix}}$. +1

Eigenvector Decomposition

1) From $\vec{x}_c$, we see $\lambda = \pm 1$ are the eigenvalues with corresponding eigenvectors $\vec{v}_+ = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

2) Build $E = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ so $E^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$.

+1

$\lambda = -1$, $\vec{v}_-$; $\lambda = 1$, $\vec{v}_+$

(I did this ordering backwards just to show it will still work!)

3) Then $\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1}\vec{f}(t) = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t \\ 0 \end{bmatrix} = \boxed{\frac{1}{2}\begin{bmatrix} -e^t \\ e^t \end{bmatrix}}$ (same result) +2

Or, you could do $\left[\begin{array}{cc|c} -1 & 1 & e^t \\ 1 & 1 & 0 \end{array}\right] \overset{R_2+R_1}{\sim} \left[\begin{array}{cc|c} -1 & 1 & e^t \\ 0 & 2 & e^t \end{array}\right] \Rightarrow \boxed{\begin{array}{l} g_1 = g_2 - e^t = -\frac{e^t}{2} \\ g_2 = e^t/2 \end{array}}$

so the eqns we need to solve are: (Note the slight difference vs morning)

$\varepsilon_1' = -\varepsilon_1 - \frac{e^t}{2} \overset{+1}{\Longleftrightarrow} \boxed{\varepsilon_1' + \varepsilon_1 = -\frac{e^t}{2}}.$

$\underline{\varepsilon_{1c}}$: $r+1=0$, $r=-1$, $\boxed{\varepsilon_{1c} = c_1 e^{-t}}$.

$\underline{\varepsilon_{1p}}$: Guess $\boxed{\varepsilon_{1p} = Ae^t}$ then, $\varepsilon_{1p}' = Ae^t$.

Plugin, $Ae^t + Ae^t = -\frac{e^t}{2}$,

$2Ae^t = -\frac{e^t}{2} \to \boxed{A = -\frac{1}{4}}$

so, $\boxed{\varepsilon_1 = c_1 e^{-t} - \frac{1}{4}e^t}$ +2

$\varepsilon_2' = \varepsilon_2 + \frac{e^t}{2} \overset{+1}{\Longleftrightarrow} \boxed{\varepsilon_2' - \varepsilon_2 = +\frac{e^t}{2}}.$

$\underline{\varepsilon_{2,c}}$: $r-1=0$, $r=1$, $\boxed{\varepsilon_{1c} = c_2 e^t}$

$\underline{\varepsilon_{2p}}$: Due to overlap, guess $\boxed{\varepsilon_{2p} = Ate^t}$

Then $\varepsilon_{2p}' = Ae^t + Ate^t$,

Plugin, $(Ae^t + \cancel{Ate^t}) - \cancel{Ate^t} = \frac{e^t}{2}$,

so $A = \frac{1}{2}$, $\varepsilon_{2p} = \frac{te^t}{2}$,

so $\boxed{\varepsilon_2 = c_2 e^t + \frac{1}{2}te^t}$ +2

4) Thus, $\vec{x} = \varepsilon_1\vec{v}_1 + \varepsilon_2\vec{v}_2$ // 1^{st} component uses $\lambda = -1$, $\vec{v}_-$; 2^{nd} component uses $\lambda = 1$, $\vec{v}_+$

$$\boxed{\vec{x} = \left(c_1 e^{-t} - \frac{1}{4}e^t\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \left(c_2 e^t + \frac{1}{2}te^t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$ +1

<u>Note</u>: Here it looks like $\vec{x}_p = \frac{1}{2}te^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{4}e^t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

But, $e^t\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \frac{1}{4}e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{4}e^t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

↑ Is a complementary piece!

only off by a complementary piece!