Solutions

Math 3D Quiz 8 Afternoon - June 1st

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Show all of your work. *There is a question on the back side.*

1. [8pts] Compute $\mathcal{L}^{-1}\left\{\frac{3}{s^4+s^2}\right\}$ using a convolution. You must integrate everything out.

1^{st} view as $3\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\}$

By Thm 6.3.1 $\longrightarrow$ $= 3\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

Here, $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$. +3

so we need either $\boxed{3t * \sin t}$ +1 or $\boxed{3\sin t * t}$:

$3t * \sin t = 3\int_0^t y\sin(t-y)\,dy$

By parts, use

$u = y$	$dv = \sin(t-y)\,dy$
$du = dy$	$v = +\cos(t-y)$ +1

By parts

+2 $= 3\left(y\cos(t-y)\Big|_{y=0}^{t} - \int_0^t \cos(t-y)\,dy\right)$

$= 3\left(t\cos(0) - 0 - \left(\frac{\sin(t-y)}{-1}\Big|_{y=0}^{t}\right)\right)$

$= 3\left(t + \sin(0) - \sin(t)\right)$

+2 $= \boxed{3(t - \sin t)}$ ✓

$3\sin t * t = 3\int_0^t (t-\tau)\sin(\tau)\,d\tau$

Use

$u = t-\tau$	$dv = \sin\tau\,d\tau$
$du = -d\tau$	$v = -\cos\tau$

By Parts

+2 $= 3\left(-(t-\tau)\cos(\tau)\Big|_{\tau=0}^{t} - \int_0^t \cos\tau\,d\tau\right)$

$= 3\left(-0 + t\cos(0) - \left(\sin\tau\Big|_{\tau=0}^{t}\right)\right)$

$= 3\left(t - \sin t + \sin(0)\right)$

+2 $= \boxed{3(t-\sin t)}$ ✓

$\longleftrightarrow$ consistent ☺

2. (a) [5pts] "Solve" (Find the Impulse Response for) $x'' + 6x' + 9x = \delta(t)$ with $x(0) = x'(0) = 0$.

* Transform the Eqn: $s^2X(s) + 6sX(s) + 9X(s) = 1.$ +2

* Isolate $X(s)$: $X(s)\cdot(s^2+6s+9) = 1$; $X(s) = \frac{1}{s^2+6s+9}$; +1

* Inverse Transform $X(s)$: $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2}\right\}$

$x(t) = e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$\boxed{x(t) = e^{-3t}\cdot t}$ + 2

(solution cont'd)

(b) [7pts] Using (a) and convolution, solve $x'' + 6x' + 9x = e^t$ with $x(0) = x'(0) = 0$. Again, you must integrate everything out.

From (a), if the Impulse Response (when RHS is $\delta(t)$) is $x(t) = te^{-3t}$,

Then here, the solution is given by $x(t) = te^{-3t} * e^t$, or equivalently,

$= e^t * te^{-3t}$. +3

Check: $X(s) \cdot (s^2 + 6s + 9) = \frac{1}{s-1}$ (Here, $\mathcal{L}\{e^t\} = \frac{1}{s-1}$)

so $X(s) = \frac{1}{s-1} \cdot \frac{1}{(s+3)^2} \Rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1} \cdot \frac{1}{(s+3)^2}\right\}$ = By Thm 6.3.1,

So we need to compute a convolution. Again, either order is valid:

$\boxed{1.}$ $x(t) = te^{3t} * e^t = \int_0^t \tau e^{-3\tau} e^{t-\tau} \, d\tau$

$$= e^t \int_0^t \tau e^{-4\tau} \, d\tau$$

By Parts, let $u = \tau$, $dv = e^{-4\tau} d\tau$; $du = d\tau$, $v = \frac{e^{-4\tau}}{-4}$

+2 $$= e^t \cdot \left[\frac{\tau e^{-4\tau}}{-4} \Big|_{\tau=0}^{t} + \frac{1}{4} \int_0^t e^{-4\tau} \, d\tau \right]$$

$$= e^t \cdot \left[\frac{te^{-4t}}{-4} + 0 + \left(\frac{e^{-4\tau}}{-16} \Big|_{\tau=0}^{t} \right) \right]$$

$$= e^t \cdot \left[\frac{te^{-4t}}{-4} - \frac{e^{-4t}}{16} + \frac{e^0}{16} \right]$$

$$= -\frac{te^{-3t}}{4} - \frac{e^{-3t}}{16} + \frac{e^t}{16}$$

+2

so $\boxed{x(t) = -\frac{te^{-3t}}{4} - \frac{e^{-3t}}{16} + \frac{e^t}{16}}$

Note: $x(t)$ does satisfy the Initial Conditions ✓

$\boxed{2.}$ $x(t) = e^t * te^{-3t} = \int_0^t e^y \cdot (t-y) e^{-3(t-y)} \, dy$

$$= \int_0^t (t-y) e^{-3t} \cdot e^{4y} \, dy$$

$$= e^{-3t} \int_0^t (t-y) e^{4y} \, dy.$$

Let $u = t-y$, $dv = e^{4y} dy$; $du = -dy$, $v = \frac{e^{4y}}{4}$

By Parts

+2 $$= e^{-3t} \left[\frac{(t-y)e^{4y}}{4} \Big|_{y=0}^{t} + \int_0^t \frac{e^{4y}}{4} \, dy \right]$$

$$= e^{-3t} \left[0 - \frac{te^0}{4} + \left(\frac{e^{4y}}{16} \right) \Big|_{y=0}^{t} \right]$$

$$= e^{-3t} \left[-\frac{t}{4} + \frac{e^{4t}}{16} - \frac{e^0}{16} \right],$$

+2

So $\boxed{x(t) = -\frac{te^{-3t}}{4} + \frac{e^t}{16} - \frac{e^{-3t}}{16}}$

Note: Either order of convolution is valid ✓