$\boxed{\text{Solutions}}$

Math 3D Quiz 8 Morning - June 1st

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Show all of your work. *There is a question on the back side.*

1. [8pts] Compute $\mathcal{L}^{-1}\left\{\frac{1}{s^3+5s^2}\right\}$ using a convolution. You must integrate everything out.

1st, view as $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s+5}\right\}$.

$\overset{\text{Thm 6.3.1}}{=} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$

Here, $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = e^{-5t}$. +3 for the two.

so we need $\boxed{t * e^{-5t}}$ +1 or $\boxed{e^{-5t} * t}$:

$$t * e^{-5t} = \int_0^t \tau\cdot e^{-5(t-\tau)}\,d\tau$$

$$= e^{-5t}\int_0^t \tau e^{5\tau}\,d\tau \qquad \begin{array}{c|c} u=\tau & dv = e^{5\tau}d\tau \\ \hline du = d\tau & v = \frac{e^{5\tau}}{5}\end{array}$$

+2 $$\overset{\text{By parts}}{=} e^{-5t}\left[\frac{\tau e^{5\tau}}{5}\Big|_{\tau=0}^{t} - \int_0^t \frac{e^{5\tau}}{5}\,d\tau\right]$$

$$= e^{-5t}\left[\frac{te^{5t}}{5} - \left(\frac{e^{5\tau}}{25}\Big|_{\tau=0}^{t}\right)\right]$$

$$= e^{-5t}\left[\frac{te^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25}\right]$$

+2 $$= \boxed{\frac{t}{5} - \frac{1}{25} + \frac{1}{25}e^{-5t}}$$ ✓

$\longleftrightarrow$ consistent :)

Integrate By Parts the 2nd one,

$$e^{-5t} * t = \int_0^t (t-y)e^{-5y}\,dy \qquad \begin{array}{c|c} u=y & dv = e^{-5y}dy \\ \hline du = dy & v = \frac{e^{-5y}}{-5}\end{array}$$

$$= \int_0^t te^{-5y}\,dy - \int_0^t ye^{-5y}\,dy$$

+2 $$\overset{\text{By parts}}{=} \frac{te^{-5y}}{-5}\Big|_{y=0}^{t} - \left[\frac{ye^{-5y}}{-5}\Big|_{y=0}^{t} + \int_0^t \frac{e^{-5y}}{+5}\,dy\right]$$

$$= -\frac{te^{-5t}}{5} + \frac{t}{5} - \left[\frac{te^{-5t}}{-5} - 0 + \frac{e^{-5y}}{-25}\Big|_{y=0}^{t}\right]$$

$$= -\cancel{\frac{te^{-5t}}{5}} + \frac{t}{5} + \cancel{\frac{te^{-5t}}{5}} + \frac{e^{-5t}}{25} - \frac{e^0}{25}$$

+2 $$= \boxed{\frac{t}{5} + \frac{e^{-5t}}{25} - \frac{1}{25}}$$ ✓ $(e^0 = 1)$

2. (a) [5pts] "Solve" (Find the Impulse Response for) $x'' + 4x = \delta(t)$ with $x(0) = x'(0) = 0$.

* Transform the eqn: $s^2 X(s) + 4X(s) = 1$; +2

* Isolate $X(s)$: $(s^2+4)X(s) = 1$, $X(s) = \frac{1}{s^2+4}$. +1

* Inverse Transform $X(s)$: $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$$\boxed{x(t) = \frac{1}{2}\sin(2t)}$$ +2

(solution cont'd)

(b) [7pts] Using (a) and convolution, solve $x'' + 4x = t$ with $x(0) = x'(0) = 0$.
Again, you must integrate everything out.

From (a), if the Impulse Response – when RHS is $\delta(t)$ – is $x(t) = \frac{1}{2}\sin(2t)$, then here, $x(t) = \frac{1}{2}\sin(2t) * t$. +3

(or equivalently, $\frac{1}{2}\, t * \sin(2t)$).

Check: $s^2 X(s) + 4X(s) = \frac{1}{s^2}$; $X(s) = \frac{1}{s^2} \cdot \frac{1}{s^2+4}$

so $x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$ which is this ✓

Again, we can compute the convolution in either order:

1. $x(t) = \frac{1}{2}\sin(2t) * t$

$$= \frac{1}{2}\int_0^t \sin(2\tau)\cdot(t-\tau)\,d\tau$$

let $u = t-\tau$ | $dv = \sin 2\tau \, d\tau$
$du = -d\tau$ | $v = -\frac{\cos 2\tau}{2}$.

By parts

+2 $$\overset{\text{}}{=} -\frac{1}{4}(t-\tau)\cos 2\tau\Big|_{\tau=0}^{t} - \frac{1}{2}\int_0^t \frac{\cos 2\tau}{2}\,d\tau$$

$$= 0 + \frac{1}{4}t\cos(0) - \left(\frac{1}{4}\cdot\frac{\sin 2\tau}{2}\Big|_{\tau=0}^{t}\right)$$

$$= \frac{t}{4} - \frac{\sin(2t)}{8}$$

+2 so $\boxed{x(t) = \frac{t}{4} - \frac{\sin(2t)}{8}}$

Note: x does satisfy the Boundary conditions.

2. $x(t) = \frac{1}{2}\, t * \sin(2t)$

$$= \frac{1}{2}\int_0^t y \sin(2(t-y))\,dy$$

let $u = y$ | $dv = \sin(2t-2y)\,dy$
$du = dy$ | $v = -\frac{\cos(2t-2y)}{-2}$

By parts

+2 $$= \frac{1}{2}\cdot\frac{y\cos(2t-2y)}{2}\Big|_{y=0}^{t} - \frac{1}{2}\int_0^t \frac{\cos(2t-2y)}{2}\,dy$$

$$= \frac{1}{4}t\cos(0) - 0 - \left(\frac{\sin(2t-2y)}{-8}\Big|_{y=0}^{t}\right)$$

$$= \frac{t}{4} + \sin(0) - \frac{\sin(2t)}{8}$$

+2 so, $\boxed{x(t) = \frac{t}{4} - \frac{\sin(2t)}{8}}$

Note: It's same answer using either order of convolution ✓