Solutions

Math 3D Quiz 9 Afternoon - June 8th

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Show all of your work. *Please continue #2 on the back side.*

1. [8pts] Last time we saw that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ centered at $x_0 = 0$. Use this to find a Taylor series of $f(x) = \frac{x^2}{1-7x}$ centered at $x_0 = 0$. What are the Radius and Interval of Convergence?

★ Make f look like $\frac{1}{1-x}$: $f(x) = x^2 \cdot \frac{1}{1-7x} = x^2 \cdot \sum_{n=0}^{\infty} (7x)^n$.

Put x^2 into series, $\boxed{f(x) = \sum_{n=0}^{\infty} 7^n x^{n+2}}$

For ROC First: Ratio Test, $\lim_{n\to\infty} \left| \frac{7^{n+1} x^{n+3}}{7^n x^{n+2}} \right| = |7x| < 1$

so divide out 7, $|x| < \frac{1}{7}$ => $\boxed{ROC = \frac{1}{7}}$

For IOC: $x_0 = 0$ so IOC is $(x_0 - ROC, x_0 + ROC) = \boxed{\left(-\frac{1}{7}, \frac{1}{7}\right)}$.

Check Endpts: $x = -\frac{1}{7}$, $f(x) = \sum_{n=0}^{\infty} \frac{7^n}{(-7)^{n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{49}$ does not converge.

$x = \frac{1}{7}$, $f(x) = \sum_{n=0}^{\infty} \frac{7^n}{7^{n+2}} = \sum_{n=0}^{\infty} \frac{1}{49} \to \infty$.

(So we exclude endpts)

2. [12pts] Solve $y'' + (x-2)^2 y' + (x-2)y = 0$ with a power series centered at $x_0 = 2$. Find the recurrence relation and give which coefficients are constants (in other words, free). Also find the first 5 terms of the power series expression of y.

① For $x_0 = 2$, $y = \sum_{k=0}^{\infty} a_k (x-2)^k$.

So $y' = \sum_{k=1}^{\infty} k a_k (x-2)^{k-1}$ and $y'' = \sum_{k=2}^{\infty} k(k-1) a_k (x-2)^{k-2}$.

② Plug into the eqn,

$$\sum_{k=2}^{\infty} k(k-1) a_k (x-2)^{k-2} + (x-2)^2 \sum_{k=1}^{\infty} k a_k (x-2)^{k-1} + (x-2) \sum_{k=0}^{\infty} a_k (x-2)^k = 0.$$

$$\sum_{k=2}^{\infty} k(k-1)a_k(x-2)^{k-2} + \sum_{k=1}^{\infty} k a_k (x-2)^{k+1} + \sum_{k=0}^{\infty} a_k (x-2)^{k+1} = 0.$$

Reindex, $n=k-2$, $k=n+2$, n starts at 0; $n=k+1$, $k=n-1$, n starts at 2; $n=k+1$, $k=n-1$, n starts at 1

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n + \sum_{n=2}^{\infty} (n-1)a_{n-1}(x-2)^n + \sum_{n=1}^{\infty} a_{n-1}(x-2)^n = 0.$$

peel off $n=0,1$ terms

Is the largest starting power, $(x-2)^2$

peel off $n=1$ term.

$$2a_2 + 6a_3(x-2) + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2}(x-2)^n + \sum_{n=2}^{\infty} (n-1)a_{n-1}(x-2)^n$$

$$+ a_0(x-2) + \sum_{n=2}^{\infty} a_{n-1}(x-2)^n = 0$$

Combine the series',

$$\Rightarrow 2a_2 + (6a_3+a_0)(x-2) + \sum_{n=2}^{\infty} (x-2)^n \cdot \Big[(n+2)(n+1)a_{n+2} + \underbrace{(n-1)a_{n-1} + a_{n-1}}_{\text{Leaves just } n a_{n-1}}\Big] = 0$$

(3) coefficients:

* constants: $2a_2 = 0$
* $(x-2)^1$: $6a_3 + a_0 = 0$
* $(x-2)^n$, $n \geq 2$: $(n+2)(n+1)a_{n+2} + n a_{n-1} = 0.$

- a_0, a_1 = free constants
- $a_2 = 0$
- $a_3 = \dfrac{-a_0}{6}$
- $a_{n+2} = \dfrac{-n a_{n-1}}{(n+2)(n+1)}$ for $n \geq 2$

$$a_6 = \frac{-4a_3}{6\cdot5} = \frac{+4a_0}{6^2\cdot5}$$

↳ Gives $a_4 = \dfrac{-2a_1}{4\cdot3}$, $a_5 = \dfrac{-3a_2}{5\cdot4} = 0$, etc...

First Terms: $\boxed{y = a_0 + a_1(x-2) - \frac{a_0}{6}(x-2)^3 - \frac{a_1}{6}(x-2)^4 + \frac{a_0}{45}(x-2)^6 + \ldots}$

(Factoring out a_0, a_1 yields the two complementary solutions)