Solutions

Math 3D Quiz 9 Morning - June 8th

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Show all of your work. *Please continue #2 on the back side.*

1. [8pts] Last time we saw that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ centered at $x_0 = 0$. Use this to find a Taylor series of $f(x) = \frac{x}{3-x}$ centered at $x_0 = 0$. What are the Radius and Interval of Convergence?

★ Make f look like $\frac{1}{1-x}$: $f(x) = x \cdot \frac{1}{3-x} = \frac{x}{3} \cdot \frac{1}{1-\left(\frac{x}{3}\right)}$

So $f(x) = \frac{x}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \boxed{\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^{n+1}}$.

For ROC: Ratio Test, $\lim_{n\to\infty} \left| \frac{\left(\frac{x}{3}\right)^{n+2}}{\left(\frac{x}{3}\right)^{n+1}} \right| = \left|\frac{x}{3}\right| < 1$.

Multiply the 3 over, $|x| < 3$, $\boxed{ROC = 3}$.

For IOC: $x_0 = 0$ so IOC is $(x_0 - ROC, x_0 + ROC)$
$= \boxed{(-3, 3)}$

Check Endpts: $x = -3$: $\sum_{n=0}^{\infty} (-1)^{n+1}$ doesn't converge (exclude).
$x = 3$: $\sum_{n=0}^{\infty} 1 \longrightarrow \infty$, diverges (exclude).

2. [12pts] Solve $y'' + 2(x^2 - 6x + 9)y = x - 3$ with a power series centered at $x_0 = 3$. Find the recurrence relation and give which coefficients are constants (in other words, free). Also find the first 5 terms of the power series of y.

① So y has the form $y = \sum_{k=0}^{\infty} a_k (x-3)^k$ now, since $x_0 = 3$.

And $y' = \sum_{k=1}^{\infty} k a_k (x-3)^{k-1}$, $y'' = \sum_{k=2}^{\infty} k(k-1) a_k (x-3)^{k-2}$.

② Plug into eqn. 1st rewrite as $y'' + 2(x-3)^2 y = (x-3)^1$.

↳ $\sum_{k=2}^{\infty} k(k-1)a_k (x-3)^{k-2} + 2(x-3)^2 \sum_{k=0}^{\infty} a_k (x-3)^k = (x-3)^1$

$$\sum_{k=2}^{\infty} k(k-1)a_k(x-3)^{k-2} + \sum_{k=0}^{\infty} 2a_k(x-3)^{k+2} = (x-3)$$

$n=k-2$
$k=n+2$
n starts at 0

$n=k+2$
$k=n-2$
n starts at 2

Reindex,

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-3)^n + \sum_{n=2}^{\infty} 2a_{n-2}(x-3)^n = (x-3)$$

Need to peel off $n=0,1$ terms.

Is the Largest Starting Power, $(x-3)^2$.

$$2a_2 + 6a_3(x-3) + \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}(x-3)^n + \sum_{n=2}^{\infty} 2a_{n-2}(x-3)^n = (x-3).$$

Combine the series',

$$2a_2 + 6a_3(x-3) + \sum_{n=2}^{\infty}(x-3)^n \cdot \left[(n+2)(n+1)a_{n+2} + 2a_{n-2}\right] = (x-3).$$

(3) Coefficients:

* Constant: $2a_2 = 0$
* $(x-3)^1$: $6a_3 = 1$
* $(x-3)^n$, $n \geq 2$: $(n+2)(n+1)a_{n+2} + 2a_{n-2} = 0$.

- a_0, a_1 = free constants
- $a_2 = 0$
- $a_3 = 1/6$
- $a_{n+2} = \dfrac{-2a_{n-2}}{(n+2)(n+1)}$ for $n \geq 2$.

Gives $a_4 = \dfrac{-2a_0}{4\cdot 3}$, $a_5 = \dfrac{-2a_1}{5\cdot 4}$, $a_6 = \dfrac{-2a_2}{6\cdot 5} = 0$,

$a_7 = \dfrac{-2a_3}{7\cdot 6} = -\dfrac{1}{7\cdot 6\cdot 3}$, etc...

First Terms: $\boxed{y = a_0 + a_1(x-3) + \frac{1}{6}(x-3)^3 - \frac{a_0}{6}(x-3)^4 - \frac{a_1}{10}(x-3)^5 - \frac{1}{126}(x-3)^7 + \dots}$

(Factoring out a_0, a_1 gives the y_c pieces; the other terms make up y_p)