Math 3D Practice Midterm Spring 2017

April 29, 2017

1. Find an explicit solution to the following differential equation:

$$xy' + \left(x + \frac{1}{3}\right) y = e^{-2x} y^{-2}$$

Strategy: D.E. is a Bernoulli's Equation.

$$y = \left(3x^{-1}e^{-2x} + Cx^{-1}e^{-3x}\right)^{\frac{1}{3}}$$

2. Find the solution to the following differential equation (implicit okay):

$$\left(e^y + 1\right)^2 e^{-y} + \left[(e^x + 1)^3 e^{-x}\right] \frac{dy}{dx} = 0,\ y(0) = 0$$

Strategy: D.E. is separable.

$$-\left(e^y + 1\right)^{-1} = \frac{1}{2}\left(e^x + 1\right)^{-2} - \frac{5}{8}$$

3. Find the solution to the following differential equation:

$$y'' - 4y' + 13y = 0,\ y(0) = -11,\ y'(0) = 19$$

Strategy: D.E. has constant coefficients. $r = 2 \pm 3i$

$$y = -11e^{2x}\cos(3x) + \frac{41}{3}e^{2x}\sin(3x)$$

4. (a) Find the general solution to the following differential equation:

$$y''' + 12y'' + 36y' = 0$$

Strategy: D.E. has constant coefficients. $y = e^{rx}$ with $r = 0, 6, 6$.

$$y = C_1 + C_2e^{-6x} + C_3xe^{-6x}$$

(b) Use your work in part (b) to solve the following differential equation:

$$y''' + 12y'' + 36y' = 108x^2$$

Strategy: $y = y_c + y_p$. Use part (a) for y_c. Using undetermined coefficients, the form for $y_p = (Ax^2 + Bx + C)x = Ax^3 + Bx^2 + Cx$. ($y_p$ was shifted by x to avoid overlap with C_1 in y_c). Find $A = 1, B = -1, C = \frac{1}{2}$.

$$y = C_1 + C_2e^{-6x} + C_3xe^{-6x} + x^3 - x^2 + \frac{1}{2}x$$

5. (a) Find the general solution to the following differential equation:

$$x^2y'' + 7xy' + 5y = 0$$

Strategy: D.E. is a Cauchy-Euler equation. $y = x^r$ with $r = -1, -5$. Hint: your auxillary equation is NOT $r^2 + 7r + 5 = 0$. You should plug in y, y', and y'' to see its actual form.

$$y = C_1x^{-1} + C_2x^{-5}$$

(b) Use your work in part (a) to solve the following differential equation:
(Hint: Use Variation of Parameters):

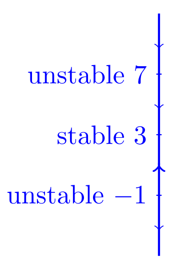
$$x^2y'' + 7xy' + 5y = 2x + x^2$$

Strategy: $y = y_c + y_p$. Use part (a) for y_c. For this problem, either variation of parameters or undetermined coefficients works (but undetermined coefficients may not work on all Cauchy-Euler problems).

$$y = C_1x^{-1} + C_2x^{-5} + \left(\frac{1}{4}x^2 + \frac{1}{12}x^3\right)x^{-1} + \left(-\frac{1}{12}x^6 - \frac{1}{28}x^7\right)x^{-5} = C_1x^{-1} + C_2x^{-5} + \frac{1}{6}x + \frac{1}{21}x^2$$

6. Find all critical points of the differential equation and plot its phase diagram:

$$\frac{dy}{dt} = -(y-7)^2(y-3)(y+1)$$



7. Determine whether the following sets of functions are linearly independent:

(a) $\{\sin x, \cos x, x\sin x\}$ L.I.

(b) $\{xe^x, x^2e^x, e^{2x}\}$ L.I.

(c) $\{\cos^2 x, \sin^2 x, 1\}$ L.D., since $\sin^2 x + \cos^2 x = 1$

8. A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of the total amount per hour. Let $W(t)$ be the number of ounces that the wizard has at time t.

   (a) Determine the first-order differential equation that models this scenario.

   Strategy: (net rate of W) = (rate of creation) - (rate of loss). The concept is similar to (net rate) = (rate in) - (rate out) in chemical concentration problems.

$$\frac{dW}{dt} = 1 - 0.05W$$

   (b) Solve your differential equation for $W(t)$ if $W(0) = 1$.

   Strategy: D.E. is both separable and linear. We can solve this problem using either method.

$$W = 20 - 19e^{\frac{t}{20}}$$