Math 3D Spring 2017 Practice Final Sketched Solutions

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Problem 1

Solve (implicit solution okay)

$$\frac{y^2+1}{x^2+1} + y\frac{dy}{dx} = 0, \quad y(0) = 0$$

Solution. When we separate the equation, we get

$$\frac{y\ dy}{y^2+1} = -\frac{1}{x^2+1}dx.$$

Integrating both sides, mainly the LHS uses a $u = y^2+1, du = 2ydy$ substitution,

$$\frac{\ln|y^2+1|}{2} = -\arctan(x) + C.$$

We can omit the absolute value because y^2+1 is positive for any y.
Solving for C, we plug in our I.C. $y(0) = 0$,

$$\frac{\ln(1)}{2} = -\arctan(0) + C \iff C = 0.$$

So, our implicit solution is

$$\frac{\ln(y^2+1)}{2} = -\arctan(x).$$

And this is all we need because an implicit form is okay :)

If you want to see the explicit solution, read on:

$$\ln(y^2+1) = -2\arctan x \iff y^2+1 = e^{-2\arctan(x)}$$

so,

$$y^2 = -1 + e^{-2\arctan(x)}$$
$$y = \pm\sqrt{-1+e^{-2\arctan(x)}}$$

(and we actually can't determine which branch of $\pm$ because y is initially 0...)
(and the domain is $-\infty < x \leq 0$ so that the exponential is bigger than 1.)

Problem 2

Find an explicit solution to

$$y' + 3y = 2xe^{-3x}.$$

Solution. Using an integrating factor of $R(x) = e^{\int 3dx} = e^{3x}$, multiply this to both sides.
Remember the LHS becomes a perfect derivative,

$$\frac{d}{dx}\left[e^{3x}y\right] = 2xe^{-3x} \cdot e^{3x}$$

which simplifies to

$$\frac{d}{dx}\left[e^{3x}y\right] = 2x.$$

Integrate both sides,

$$e^{3x}y = x^2 + C,$$

and isolating for y, we get our explicit solution,

$$y = e^{-3x}(x^2 + C).$$

Alternative Solution: Recall that we can treat these like second orders. Here, the auxiliary equation is

$$y' + 3y = 0 \iff r + 3 = 0 \iff r = -3, \quad y_c = Ce^{-3x}.$$

So $y_c = e^{-3x}$. The RHS inhomogeneity is $2xe^{-3x}$ which comes from this root actually. (For example, taking a derivative, it has a e^{-3x} piece overlapping with y_c). Thus, we would have to shift our guess once (because the root has multiplicity 1), from guessing $y_p = Axe^{-3x}$ to Ax^2e^{-3x}. Then

$$y_p = Ax^2e^{-3x} \longrightarrow y_p' = 2Axe^{-3x} - 3Ax^2e^{-3x}.$$

Plugging in,

$$y_p' + 3y_p = 2xe^{-3x} \longrightarrow (2Axe^{-3x} - 3Ax^2e^{-3x}) + 3Ax^2e^{-3x} = 2xe^{-3x}$$

$$\iff 2Axe^{-3x} = 2xe^{-3x} \iff A = 1.$$

So with $A = 1$,

$$y_p = x^2e^{-3x}$$

and the general solution is given by $y = y_c + y_p$,

$$y = Ce^{-3x} + x^2e^{-3x}.$$

(This is the same answer we got with integration factor).

Problem 3

General solution to

$$y'' + y' + y = \sin x.$$

Solution. First solve for y_c with the Auxiliary (Characteristic) equation,

$$r^2 + r + 1 = 0 \iff r = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

which means

$$y_c = C_1e^{-x/2}\cos(\sqrt{3}x/2) + C_2e^{-x/2}\sin(\sqrt{3}x/2).$$

I'd highly recommend doing undetermined coefficients - you can see that the derivatives of y_c are bad for setting up variation of parameters.

With undetermined coefficients, the RHS is just $\sin(x)$ so this means we should guess

$$y_p = A\sin x + B\cos x.$$

Don't forget, we have to include both sine and cosine! Then,

$$y_p' = A\cos x - B\sin x, \qquad y_p'' = -A\sin x - B\cos x.$$

Plugging in,

$$y_p'' + y_p' + y_p = \sin x \longrightarrow (-A\sin x - B\cos x) + (A\cos x - B\sin x) + (A\sin x + B\cos x) = \sin x.$$

Reducing a bit,

$$A\cos x - B\sin x = \sin x.$$

Reading off the coefficients of the function types $\cos x, \sin x$, we have

$$A = 0, \quad B = -1.$$

Therefore

$$y_p = -\cos x.$$

And, $y = y_c + y_p$ for the general solution, so

$$y = C_1 e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}\,x\right) + C_2 e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}\,x\right) - \cos x.$$

Variation of Parameters Setup If Interested: It's pretty bad, the system we have to solve is

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = f(x) \end{cases} \longrightarrow \begin{cases} u_1'e^{-x/2}\cos(\sqrt{3}x/2) + u_2'e^{-x/2}\sin(\sqrt{3}x/2) = 0 \\ u_1' \cdot \frac{d}{dx}[e^{-x/2}\cos(\sqrt{3}x/2)] + u_2' \cdot \frac{d}{dx}[e^{-x/2}\sin(\sqrt{3}x/2)] = \sin(x) \end{cases}$$

in which ... spare me the work of solving that ...

Problem 4

Find a real-valued vector (general) solution to

(a)

$$\vec{x}'(t) = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \vec{x}(t).$$

Solution. First find the eigenvalues:

$$\det \begin{bmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

The roots of this, either by quadratic equation or factoring, are $\lambda = 3$ (multiplicity 2).

Now let's find eigenvectors, solving $(A - 3I)\vec{v} = \vec{0}$,

$$\begin{bmatrix} -2 & -2 & \vdots\, 0 \\ 2 & 2 & \vdots\, 0 \end{bmatrix} \to \begin{matrix} R_2 = -R_1, \\ 2v_1 + 2v_2 = 0 \end{matrix} \to \begin{matrix} v_1 = -v_2 \\ v_2 = free \end{matrix} \to \vec{v} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We only get one eigenvector of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $\lambda = 3$ is defective once.

We need a generalized eigenvector $\vec{w}$ by solving for $(A - 3I)\vec{w} = \vec{v}$,

$$\begin{bmatrix} -2 & -2 & \vdots & -1 \\ 2 & 2 & \vdots & 1 \end{bmatrix} \to \begin{matrix} R_2 = -R_1, \\ 2w_1 + 2w_2 = 1 \end{matrix} \to \begin{matrix} w_1 = -w_2 + \frac{1}{2} \\ w_2 = free \end{matrix} \to \vec{w} = w_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}.$$

The generalized eigenvector we get is $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$.

Now we can construct the general solution for (a),

$$\text{For (a):} \qquad \vec{x}(t) = C_1 e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{3t} \left(\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + \frac{t}{1!} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right).$$

(b)

$$\vec{x}'(t) = \begin{bmatrix} 4 & -1 \\ 13 & 0 \end{bmatrix} \vec{x}(t).$$

Hint: One eigenvalue and its eigenvector are $\lambda_1 = 2 + 3i, \quad \vec{v}_1 = \begin{bmatrix} 2 + 3i \\ 13 \end{bmatrix}$.

Solution. Using the hint you can just skip the steps of finding the eigenvalues and eigenvectors. Here's the verification of the hint (feel free to skip):

If you solve for the eigenvalues, what you'll get are $\lambda = 2 \pm 3i$ so we were just given one of the pairs and its corresponding eigenvector.

$$\det \begin{bmatrix} 4 - \lambda & -1 \\ 13 & -\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 13, \qquad \lambda = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i.$$

If we use the positive one, and solve for an eigenvector, we'd solve $(A - (2 + 3i)I)\vec{v} = 0$,

$$\begin{bmatrix} 2 - 3i & -1 & \vdots\, 0 \\ 13 & -2 - 3i & \vdots\, 0 \end{bmatrix}.$$

Recall, we know for a 2 by 2 that these rows must be (complex scalar) multiples of each other (in this case, $R_2 = (2 + 3i)R_1$ for example) so we can just pick out what one of the rows says, e.g. Row 2,

$$13v_1 - (2 + 3i)v_2 = 0 \iff v_1 = \frac{(2 + 3i)}{13} v_2.$$

Set the one with the complex number to be free, so here, set $v_2 = free$, and rescale by 13,

$$\begin{array}{c} v_1 = \frac{2+3i}{13} v_2 \\ v_2 = free \end{array} \longrightarrow \vec{v} = v_2 \begin{bmatrix} \frac{2+3i}{13} \\ 1 \end{bmatrix} = \tilde{v}_2 \begin{bmatrix} 2+3i \\ 13 \end{bmatrix}.$$

So this is what the hint gave us. Good. :)

Start here if just using the hint: Now using this, all we need to do is take one such complex solution and split it into its Real and Imaginary Parts. We know one such solution would look like

$$e^{\lambda_1 t}\vec{v}_1 = e^{(2+3i)t} \begin{bmatrix} 2+3i \\ 13 \end{bmatrix}.$$

I highly recommend splitting the exponent and the vector into real and imaginary parts.
Using that $e^{(2+3i)t} = e^{2t+i3t} = e^{2t} \cdot e^{i3t}$ and using Euler's identity, $e^{i3t} = \cos(3t) + i\sin(3t)$,

$$\begin{aligned} e^{\lambda_1 t}\vec{v}_1 &= e^{2t} \cdot e^{i3t} \begin{bmatrix} 2+3i \\ 13 \end{bmatrix} \\ &= e^{2t}(\cos(3t) + i\sin(3t)) \left(\begin{bmatrix} 2 \\ 13 \end{bmatrix} + i \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right) \end{aligned}$$

Foil everything,

$$= e^{2t} \left(\begin{bmatrix} 2\cos 3t \\ 13\cos 3t \end{bmatrix} + i \begin{bmatrix} 3\cos 3t \\ 0 \end{bmatrix} + i \begin{bmatrix} 2\sin 3t \\ 13\sin 3t \end{bmatrix} + i^2 \begin{bmatrix} 3\sin 3t \\ 0 \end{bmatrix} \right)$$

Recall $i^2 = -1$,

$$= e^{2t} \begin{bmatrix} 2\cos 3t - 3\sin 3t \\ 13\cos 3t \end{bmatrix} + i \cdot e^{2t} \begin{bmatrix} 2\sin 3t + 3\cos 3t \\ 13\sin 3t \end{bmatrix}$$

So we see that the Real (Re) and Imaginary (Im) parts are:

$$Re : e^{2t} \begin{bmatrix} 2\cos 3t - 3\sin 3t \\ 13\cos 3t \end{bmatrix}, \quad Im : e^{2t} \begin{bmatrix} 2\sin 3t + 3\cos 3t \\ 13\sin 3t \end{bmatrix}$$

dropping the i in the imaginary part! Then, we use these to build our general solution, for (b),

$$\vec{x}(t) = C_1(Real\ Part) + C_2(Imaginary\ Part),$$

$$\text{For (b)}: \quad \vec{x}(t) = C_1 e^{2t} \begin{bmatrix} 2\cos 3t - 3\sin 3t \\ 13\cos 3t \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2\sin 3t + 3\cos 3t \\ 13\sin 3t \end{bmatrix}$$

(c)

$$\vec{x}'(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

Solution. Even with inhomogeneous equations, we should still get the complementary solution (and we have to if we want the general solution, which we do here). We need to do it to see if we can do Eigenvector Decomposition, and we need it for $\vec{x}_c$ for Undetermined Coefficients.

Here, the matrix is (upper) triangular - this means the eigenvalues lie on the main diagonal (check the characteristic equation if you're skeptical). So, $\lambda = 1, 2$.

The corresponding eigenvectors are:

(Solve for $(A-I)\vec{v}=0$ and $(A-2I)\vec{w}=0$)

$$\lambda = 1: \begin{bmatrix} 0 & 1 & \vdots\, 0 \\ 0 & 1 & \vdots\, 0 \end{bmatrix} \quad \to \quad \begin{matrix} v_1 = free \\ v_2 = 0 \end{matrix} \quad \to \vec{v} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is our eigenvector for } \lambda = 1.$$

$$\lambda = 2: \begin{bmatrix} -1 & 1 & \vdots\, 0 \\ 0 & 0 & \vdots\, 0 \end{bmatrix} \quad \to \quad \begin{matrix} w_1 = w_2 \\ w_2 = free \end{matrix} \quad \to \vec{w} = w_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ is our eigenvector for } \lambda = 2.$$

So now, you can choose either Eigenvector Decomposition or Undetermined Coefficients.

Undetermined Coefficients: From this, we see that $\vec{x}_c = C_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For the inhomogeneous part, $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, it's just a constant vector - it does not overlap with any of our complementary solution, so we guess a constant vector to be our particular solution,

$$\vec{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \quad \text{Then } \vec{x}_p{}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Plugging this in for $\vec{x}_p{}' = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}_p + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ 2a_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} \iff \begin{matrix} a_1 + a_2 = -4 \\ 2a_2 = -3 \end{matrix}$$

So we see from the 2nd row that $a_2 = -3/2$. Plugging that into the 1st row, $a_1 - 3/2 = -4 \iff a_1 = -5/2$. So this means

$$\vec{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -3/2 \end{bmatrix}$$

and hence our general solution is $\vec{x} = \vec{x}_c + \vec{x}_p$, so for part (c),

$$\text{For part (c) Undet. Coefs:} \qquad \vec{x}(t) = C_1 e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5/2 \\ -3/2 \end{bmatrix}.$$

Eigenvector Decomposition: From the eigenvectors we got, we build our diagonalization matrix E by

$$E = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Keep in mind this is in the order of 1st column $\sim \lambda = 1$ *and 2nd column* $\sim \lambda = 2$ *!!*

Now we need to rewrite our inhomogeneous part in our eigenbasis by solving $E\vec{g}(t) = \vec{f}(t)$, so here,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

You can read that $g_2(t) = 3$ so then $g_1(t) + g_2(t) = 4$ gives us $g_1(t) = 1$.

An equivalent way to do this, compute $\vec{g}(t) = E^{-1}\vec{f}(t)$ which here is

$$\begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = \frac{1}{1-0}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad g_1(t) = 1,\ g_2(t) = 3.$$

Either way we get $g_1(t) = 1$ and $g_2(t) = 3$. They are constant *functions.*

Now we build the system using what our eigenvalues and g's are - *Remember the ordering!!* -

$$\begin{cases} \xi_1' = \lambda_1\xi_1 + g_1(t) \\ \xi_2' = \lambda_2\xi_2 + g_2(t) \end{cases} \longrightarrow \begin{cases} \xi_1' = \xi_1 + 1 \\ \xi_2' = 2\xi_2 + 3 \end{cases} \longrightarrow \begin{cases} \xi_1' - \xi_1 = 1 \\ \xi_2' - 2\xi_2 = 2 \end{cases}$$

And we solve for ξ_1, ξ_2 by either integration factor (Ch 1.4) or treating them like 2nd order equations (Ch 2.5). Let's do one of each just to show both ways:

1) $\xi_1' - \xi_1 = 1$: The Auxiliary equation is $r - 1 = 0,\ r = 1$ so $\xi_{1,c} = C_1e^t$.
The RHS function does not overlap with this so we'd just guess $\xi_{1,p} = A$ a constant. Then $\xi_{1,p}' = 0$.
Plugging in, $\xi_{1,p}' - \xi_{1,p} = 1 \iff -A = 1$ so $A = -1$ and $\xi_{1,p} = -1$. This gives us

$$\xi_1 = C_1e^t - 1. \qquad (\text{From } \xi_1 = \xi_{1,c} + \xi_{1,p}).$$

2) $\xi_2' - 2\xi_2 = 3$: Let us use an integrating factor $R(t) = e^{\int -2dt} = e^{-2t}$.
Multiplying this to both sides,

$$\frac{d}{dt}[e^{-2t}\xi_2] = 3e^{-2t} \overset{integrate}{\longrightarrow} e^{-2t}\xi_2 = \frac{3}{-2}e^{-2t} + C_2.$$

Divide by the e^{-2t}, in other words multiply by e^{2t}, to isolate ξ_2,

$$\xi_2 = C_2e^{2t} - \frac{3}{2}.$$

Now we build our solution, $\vec{x} = \xi_1\vec{v}_1 + \xi_2\vec{v}_2$ where $\vec{v}_1$ is the eigenvector for $\lambda = 1$ and $\vec{v}_2$ is the eigenvector for $\lambda = 2$ (*again, remember the ordering!*),

$$\text{For part(c) EigenDecomp:} \qquad \vec{x} = \left(C_1e^t - 1\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \left(C_2e^{2t} - \frac{3}{2}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note: In both methods, we get an overall constant of $-5/2$ in the 1st entry and a $-3/2$ in the 2nd entry. They're indeed the same answer, as they should be :)

Problem 5

Determine if the set is linearly independent:

$$\left\{ e^{-t}\begin{bmatrix} 1 \\ -6 \end{bmatrix},\ e^{-2t}\begin{bmatrix} 1 \\ -6 \end{bmatrix},\ e^{-t}\begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\}.$$

Solution. We setup the definition of linear independence first,
we want to solve for c_1, c_2, c_3 such that for *all times* t,

$$c_1 e^{-t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The smallest exponent is e^{-2t}, so if we divide by this and instead take the limit as $t \to -\infty$ (as opposed to $+\infty$), we see

$$\lim_{t \to -\infty} \left(c_1 e^{t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right) = c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff c_2 = 0.$$

So we get c_2 must be 0.

We now have to just solve for c_1, c_3 from plugging in $c_2 = 0$,

$$c_1 e^{-t} \begin{bmatrix} 1 \\ -6 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now, let us plug in $t = 0$, or just divide out the e^{-t} common to both terms, we have

$$c_1 \begin{bmatrix} 1 \\ -6 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 1 & 2 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

You can row reduce and get that $c_1 = c_3 = 0$.

Another slick way is that the determinant of this 2 by 2 matrix is not zero (determinant is 9) which means it is invertible and the only solution to 0 is the 0 vector, i.e. $\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is the only solution. Anyways in either case, we deduce $c_1 = c_3 = 0$.

We had $c_2 = 0$ already.

We see that all constants c_1, c_2, c_3 must equal 0 for the zero-equation to hold for all times t, and since the only solution is $c_1 = c_2 = c_3 = 0$, the trivial solution, the set is linearly independent.

Problem 6

Solve the forwards and inverse Laplace Transforms:

(a)

$$\mathcal{L}^{-1} \left\{ \frac{1}{4s^2 + 1} \right\}$$

Solution. This one looks almost like the one for $\sin(\omega x)$ but it's got a coefficient of 4 in front of s^2. So we need to manipulate this a little. Mainly to get rid of the 4, factor it out of everything in the denominator,

$$\mathcal{L}^{-1} \left\{ \frac{1}{4s^2 + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{4} \cdot \frac{1}{(s^2 + 1/4)} \right\} = \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1/4} \right\}$$

This is almost the one for sine, except for the scaling on top. Here $\omega^2 = 1/4$ so $\omega = 1/2$,

$$\begin{aligned}\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1/4}\right\} &= \frac{1}{4}\cdot\frac{1}{1/2}\sin(t/2)\\ &= \frac{1}{2}\sin(t/2).\end{aligned}$$

So,

$$\text{Answer (a):} \qquad \mathcal{L}^{-1}\left\{\frac{1}{4s^2+1}\right\} = \frac{1}{2}\sin\left(\frac{t}{2}\right).$$

(b)

$$\mathcal{L}\left\{(t-3)^3[u(t-1)-u(t-3)]\right\}$$

Solution. Since we have Heavisides, we're going to have to make our function in front have the same shifts as the Heavisides. We're going to break it up first as

$$\mathcal{L}\left\{(t-3)^3[u(t-1)-u(t-3)]\right\} = \mathcal{L}\left\{(t-3)^3u(t-1)\right\} - \mathcal{L}\left\{(t-3)^3u(t-3)\right\}$$

And let's find each piece. The 2nd one is perfetly a shifting property,

$$\mathcal{L}\left\{(t-3)^3u(t-3)\right\} = e^{-3s}\mathcal{L}\{t^3\} = e^{-3s}\cdot\frac{3!}{s^4} = \frac{6e^{-3s}}{s^4}.$$

The 1st one requires reshifting. We need to rewrite it as

$$\begin{aligned}\mathcal{L}\left\{(t-3)^3u(t-1)\right\} &= \mathcal{L}\left\{((t-1)+1-3)^3u(t-1)\right\}\\ &= \mathcal{L}\left\{((t-1)-2)^3u(t-1)\right\}\\ &= e^{-s}\mathcal{L}\left\{(t-2)^3\right\}\\ &= e^{-s}\mathcal{L}\left\{t^3-6t^2+12t-8\right\}\\ &= e^{-s}\cdot\left(\frac{3!}{s^4}-6\frac{2!}{s^3}+12\frac{1}{s^2}-\frac{8}{s}\right).\end{aligned}$$

So, Answer (b):

$$\begin{aligned}\mathcal{L}\left\{(t-3)^3[u(t-1)-u(t-3)]\right\} &= \mathcal{L}\left\{(t-3)^3u(t-1)\right\} - \mathcal{L}\left\{(t-3)^3u(t-3)\right\}\\ &= e^{-s}\cdot\left(\frac{6}{s^4}-\frac{12}{s^3}+\frac{12}{s^2}-\frac{8}{s}\right)-\frac{6e^{-3s}}{s^4}.\end{aligned}$$

(c)

$$\mathcal{L}^{-1}\left\{\frac{5}{(s+1)^6((s+2)^2+25)}\right\}$$

and only for (c) we can use convolution.

Solution. First pulling out the constant 5, we can rewrite this as a product of transforms,

$$\mathcal{L}^{-1}\left\{\frac{5}{(s+1)^6((s+2)^2+25)}\right\} = 5\ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^6}\cdot\frac{1}{(s+2)^2+25}\right\}$$

The inverse transform of each piece is given as

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^6}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} = e^{-t}\cdot\frac{t^5}{5!}$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+25}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{e^{-2t}\sin(5t)}{5}.$$

So, recalling that $\mathcal{L}\{f*g\} = F(s)\cdot G(s)$,

Answer (c):

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{5}{(s+1)^6((s+2)^2+25)}\right\} &= 5\ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^6}\cdot\frac{1}{(s+2)^2+25}\right\}\\ &= 5\cdot\left(\frac{t^5e^{-t}}{5!} * \frac{e^{-2t}\sin(5t)}{5}\right)\end{aligned}$$

where the * is the convolution.

Problem 7

Use Laplace Transform to solve these initial value problems:
(a)

$$x'' + 4x' + 3x = 1,\quad x(0) = 0,\ x'(0) = 2.$$

Solution. First transform the equation,

$$(s^2X(s) - 2) + 4sX(s) + 3X(s) = \frac{1}{s}.$$

Next isolate $X(s)$,

$$X(s)\cdot(s^2+4s+3) = \frac{1}{s} + 2 = \frac{1+2s}{s}.$$

Note $s^2+4s+3 = (s+1)(s+3)$,

$$X(s) = \frac{1+2s}{s(s+1)(s+3)}.$$

We decompose this with partial fractions as

$$\frac{1+2s}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$\iff 1 + 2s = A(s+1)(s+3) + Bs(s+3) + Cs(s+1).$$

If we plugin good values of s, we can just get A, B, C:

$$\begin{aligned} s = 0: \quad 1 &= A(1)(3) + 0 + 0 \iff A = 1/3. \\ s = -1: -1 &= 0 + B(-1)(2) + 0 \iff B = 1/2. \\ s = -3: -5 &= 0 + 0 + C(-3)(-2) \iff C = -5/6. \end{aligned}$$

So, plugging in A, B, C,

$$X(s) = \frac{1/3}{s} + \frac{1/2}{s+1} - \frac{5/6}{s+3}.$$

Inverse transforming,

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1/3}{s} + \frac{1/2}{s+1} - \frac{5/6}{s+3}\right\} \\ x(t) &= \frac{1}{3} + \frac{1}{2}e^{-t} - \frac{5}{6}e^{-3t}. \end{aligned}$$

(Note this is what we should get using Chapter 2 methods).

(b)

$$x'' + x = \cos(3t), \quad x(0) = 1, \; x'(0) = 0.$$

Solution. First transform the equation,

$$(s^2 X(s) - s) + X(s) = \frac{s}{s^2+9}.$$

Isolate $X(s)$,

$$\begin{aligned} X(s) \cdot (s^2 + 1) &= s + \frac{s}{s^2+9} \\ X(s) &= \frac{s}{s^2+1} + \frac{s}{s^2+9} \cdot \frac{1}{s^2+1}. \end{aligned}$$

Now inverse transform, I used convolution on the 2nd term (you could use partial fractions too),

$$x(t) = \cos(t) + \cos(3t) * \sin(t).$$

We need to compute that convolution, using that $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$,

$$\begin{aligned} \cos(3t) * \sin(t) &= \int_0^t \cos(3y)\sin(t-y)dy \\ &= \frac{1}{2}\int_0^t [\sin(t-4y) + \sin(t+2y)]dy \\ &= \frac{1}{2}\left[+\frac{\cos(t-4y)}{+4} - \frac{\cos(t+2y)}{2}\right]_{y=0}^{t} \\ \text{(recall cosine is even)} &= \frac{1}{2}\left[\frac{\cos(-3t) - \cos(t)}{4} - \frac{\cos(3t) - \cos(t)}{2}\right] \\ \text{(so } \cos(3t) = \cos(-3t)\text{)} \quad &= \frac{1}{2}\left[-\frac{\cos(3t) + \cos(t)}{4}\right] \end{aligned}$$

so overall,

$$\begin{aligned} x(t) &= \cos(t) + \cos(3t) * \sin(t) \\ &= \cos(t) + \frac{1}{8}\left(\cos(t) - \cos(3t)\right). \end{aligned}$$

Note: If you use partial fractions on the 2nd term, it becomes

$$\frac{s}{s^2+9} \cdot \frac{1}{s^2+1} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+1} = \frac{-s/8}{s^2+9} + \frac{s/8}{s^2+1}$$

which inverse transforms to

$$-\frac{1}{8}\cos(3t) + \frac{1}{8}\cos(t)$$

which is the same as what we got using the convolution.

(c)

$$x'' + 2x' + x = \delta(t-3), \quad x(0) = 0, \ x'(0) = 1.$$

Solution. Transforming the equation,

$$(s^2 X(s) - 1) + 2sX(s) + X(s) = e^{-3s}.$$

Isolating $X(s)$,

$$\begin{aligned} X(s) \cdot (s^2 + 2s + 1) &= e^{-3s} + 1 \\ X(s) &= \frac{1}{(s+1)^2} + \frac{e^{-3s}}{(s+1)^2}. \end{aligned}$$

To inverse transform this, we need to use both shifting properties,

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\} \\ &= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + e^{-t}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\} \\ &= te^{-t} + \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\}. \end{aligned}$$

Let us compute $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\}$ on its own. First, without the e^{-3s},

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = te^{-t}.$$

Therefore, using the 2nd shifting property, this is our little $f(t)$. We get,

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+1)^2}\right\} = u(t-3)f(t-3) = u(t-3) \cdot (t-3)e^{-(t-3)}.$$

So we use this for $x(t)$,

$$x(t) = te^{-t} + u(t-3) \cdot (t-3)e^{-(t-3)}.$$

Problem 8

Solve using power series centered at the given x_0. Find the recurrence relation and which coefficients are constants.

(a)

$$y'' + x^2 y = 5 + x, \quad x_0 = 0.$$

Solution. About $x_0 = 0$, we assume that y has the form

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad y' = \sum_{k=1}^{\infty} k a_k x^{k-1}, \quad y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

Plugging these into the equation,

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x^2 \sum_{k=0}^{\infty} a_k x^k = 5 + x.$$

merge the x^2 into the 2nd sum,

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^{k+2} = 5 + x.$$

Now we first have to reindex everything to align at x^n.
For the 1st series, $n = k - 2$ so $k = n + 2$ and n starts at 0 (since k started at 2).
For the 2nd series, $n = k + 2$ so $k = n - 2$ and n starts at 2 (since k started at 0).

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 5 + x.$$

The 2nd series starts at the highest power at $n = 2$ with x^2, so from the 1st series, we need to peel off the $n = 0$ and $n = 1$ terms (the x^0, x^1 terms),

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 5 + x.$$

Now we can combine the series,

$$2a_2 + 6a_3 x + \sum_{n=2}^{\infty} x^n \cdot \Big[(n+2)(n+1) a_{n+2} + a_{n-2} \Big] = 5 + x.$$

Matching the coefficients of powers of x^n now, since we have a 2nd order, we'll see a_0, a_1 are free constants (they would be determined by Initial Conditions). Then from there we read off coefficients in powers of x^n. Note that the RHS is just zero for powers starting at x^2.

$$\begin{cases} a_0, a_1 = free\ constants \\ Constant,\ x^0 : \ 2a_2 = 5 \iff a_2 = 5/2 \\ x^1 : \qquad 6a_3 = 1 \iff a_3 = 1/6 \\ x^n : \qquad (n+2)(n+1) a_{n+2} + a_{n-2} = 0 \iff a_{n+2} = \frac{-a_{n-2}}{(n+2)(n+1)} \quad \text{for} \quad n \geq 2. \end{cases}$$

This is all we need for the problem. The Recursive Relation gives us $a_4, a_5, ...$ and on.

We don't have to, but to see the first few terms,

$$\begin{cases} a_0, a_1 = free\ constants \\ a_2 = 5/2 \\ a_3 = 1/6 \\ a_4 = \frac{-a_0}{4\cdot 3} \\ a_5 = \frac{-a_1}{5\cdot 4} \\ a_6 = \frac{-a_2}{6\cdot 5} = -\frac{5}{2\cdot 6\cdot 6} \\ a_7 = \frac{-a_3}{7\cdot 6} = -\frac{1}{7\cdot 6\cdot 6} \\ a_8 = \frac{-a_4}{8\cdot 7} = \frac{a_0}{8\cdot 7\cdot 4\cdot 3} \\ \vdots \end{cases}$$

and the first few terms of y are

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{5}{2}x^2 + \frac{1}{6}x^3 - \frac{a_0}{12}x^4 - \frac{a_1}{20}x^5 - \frac{5}{72}x^6 - \frac{1}{252}x^7 + \frac{a_0}{672}x^8 \\ y(x) &= a_0\left(1 - \frac{x^4}{12} + \frac{x^8}{672} - ...\right) + a_1\left(x - \frac{x^5}{20} + ...\right) + \left(\frac{5x^2}{2} + \frac{x^3}{6} - \frac{5x^6}{72} + ...\right) \\ &= \qquad\qquad y_c \qquad\qquad + \qquad y_p \end{aligned}$$

where the 1st two pieces attached to a_0, a_1 we can infer are part of the complementary solution, the last is the particular solution.

(b)

$$y'' - (x+1)y' - y = 0, \quad x_0 = -1.$$

Solution. Now that $x_0 = -1$, our solution uses powers of $(x+1)^k$ and is of the form

$$y = \sum_{k=0}^{\infty} a_k (x+1)^k, \quad y' = \sum_{k=1}^{\infty} k a_k (x+1)^{k-1}, \quad y'' = \sum_{k=2}^{\infty} k(k-1) a_k (x+1)^{k-2}.$$

Plugging these into the equation,

$$\sum_{k=2}^{\infty} k(k-1)a_k(x+1)^{k-2} - (x+1)\sum_{k=1}^{\infty} k a_k (x+1)^{k-1} - \sum_{k=0}^{\infty} a_k (x+1)^k = 0.$$

We can pull in the whole $(x+1)$ term attached to y' in the middle!

$$\sum_{k=2}^{\infty} k(k-1)a_k(x+1)^{k-2} - \sum_{k=1}^{\infty} k a_k (x+1)^{k} - \sum_{k=0}^{\infty} a_k (x+1)^k = 0.$$

We first have to reindex everything to align at $(x-1)^n$.
For the 1st series, $n = k-2$ so $k = n+2$ and n starts at 0 (since k started at 2).
For the 2nd series, $n = k$ so $k = n$ and n starts at 1 (since k started at 1).

For the 3rd series, $n = k$ so $k = n$ and n starts at 0 (since k started at 0).

$$\text{Reindexed:} \quad \sum_{n=0}^{\infty}(n+2)(n+1)(x+1)^n - \sum_{n=1}^{\infty} na_n(x+1)^n - \sum_{n=0}^{\infty} a_n(x+1)^n = 0.$$

The 2nd series starts at the highest power at $n = 1$ with $(x+1)^1$, so from the 1st and 3rd series, we need to peel off the $n = 0$ term (the $(x+1)^0$ terms) from each,

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x+1)^n - \sum_{n=1}^{\infty} na_n(x+1)^n - a_0 - \sum_{n=1}^{\infty} a_n(x+1)^n = 0.$$

Combining the sums now, and moving the $(-a_0)$ to the front,

$$2a_2 - a_0 + \sum_{n=1}^{\infty}(x+1)^n \cdot \Big[(n+2)(n+1)a_{n+2} - (n+1)a_n\Big] = 0.$$

Now do the same thing matching coefficients of $(x+1)^n$ powers this time with the RHS = 0. Again, since our equation is 2nd order, we'll see a_0, a_1 are free constants.

$$a_0, a_1 = \mathit{free\ constants}$$
$$\mathit{Constant},\ (x+1)^0 : 2a_2 - a_0 = 0 \iff a_2 = a_0/2.$$
$$(x+1)^n : (n+2)(n+1)a_{n+2} - (n+1)a_n = 0 \iff a_{n+2} = \frac{a_n}{n+2}, \quad \text{for } n \geq 1.$$

Again, this is all we need for the problem. The Recursive Relation gives us $a_3, a_4, ...$ and on.

We don't have to, but to see the first few terms,

$$\begin{cases} a_0, a_1 = \mathit{free\ constants} \\ a_2 = a_0/2 \\ a_3 = a_1/3 \\ a_4 = a_2/4 = a_0/8 \\ a_5 = a_1/5 = a_1/15 \\ a_6 = a_4/6 = a_0/48 \\ a_7 = a_5/7 = a_1/105 \\ \vdots \end{cases}$$

so the first few terms of y are

$$y(x) = a_0 + a_1(x+1) + \frac{a_0}{2}(x+1)^2 + \frac{a_1}{3}(x+1)^3 + \frac{a_0}{8}(x+1)^4 + \frac{a_1}{15}(x+1)^5$$
$$+ \frac{a_0}{48}(x+1)^6 + \frac{a_1}{105}(x+1)^7 + ...$$
$$y(x) = a_0\left(1 + \frac{(x+1)^2}{2} + \frac{(x+1)^4}{8} + ...\right) + a_1\left(x + \frac{(x+1)^3}{3} + \frac{(x+1)^5}{15} + ...\right)$$

which we can think of like

$$\sim \qquad c_1y_1 \qquad + \qquad c_2y_2$$

which are like the two complementary pieces we'd get for the general solution.