Ex. Show $\{\underset{f(x)}{\cos 2x}, \underset{g(x)}{\cos^2 x}, \underset{h(x)}{\sin^2 x}\}$ is Linearly Dependent. (L.D.)

Method 1: In class, saw $\cos 2x = \cos^2 x - \sin^2 x$, $f(x) = g(x) - h(x)$,
In other words, it is a Linear Combination (L.C.) of the other 2 functions, hence, the set is L.D.
[The nontrivial solution to 0 is $\cos 2x - \cos^2 x + \sin^2 x = 0$, $f(x) - g(x) + h(x) = 0$.]

More Ideal, I suppose?

(If you can't see the L.C. in method 1)

Method 2: Plug & chug definition, $c_1 \cos 2x + c_2 \cos^2 x + c_3 \sin^2 x = 0$.

Plug in "good" pts,

$x = 0 \rightsquigarrow c_1 + c_2 = 0.$

★ Note: $x = \pm\pi$ gets this eqn again so avoid repeating same eqns.

$x = \frac{\pi}{2} \rightsquigarrow -c_1 + c_3 = 0.$

$x = \frac{\pi}{4} \rightsquigarrow \frac{c_2}{2} + \frac{c_3}{2} = 0.$ Rescale by 2.

[Note: $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$]

As matrix,
$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

★ $R_2$ and $R_3$ are same rows!

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

★ Now Backsolve, in col. 2,

$$\xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} c_1 = c_3 \\ c_2 = -c_3 \\ c_3 = \text{free} \end{array}}$$

★ This is saying $c_3 \cos 2x - c_3 \cos^2 x + c_3 \sin^2 x = 0$ for any $c_3$, (nonzero in particular)

And this is true because $c_3(\underbrace{\cos^2 x - \sin^2 x}_{\cos 2x} - \cos^2 x + \sin^2 x) = 0$ for any $c_3$.

Thus, we don't need all coefs equal to zero $\Rightarrow$ Set is L.D.