Project description

There are many important engineering and geophysical problems in which rotation significantly modifies the properties of fluid flows(see Pedlosky [45], for mathematical aspects Lions, Temam and Wang [39]). In particular, large-scale atmospheric and oceanic flows are dominated by rotational effects and the impact of shallowness on the large scale dynamics. Another important factor in the dynamics of oceans and the atmosphere is stratification. Long-time computation of 3D geophysical flows using unmodified Euler equations is prohibitive due to severe accuracy and time step restrictions, since the time step has to be small with respect to the period of the rotation. This makes it very important to develope new analytic methods which give the possibility to study and, if possible, to solve the equations.

The impact of the rotation on the dynamics is through inertial waves generated by the rotation; the waves interact thanks to the bilinear term. Important role of nonlinear wave interactions is well-known in physics and mathematics. From a formal point of view, wave interactions are similar in hydrodynamics and optics, they are studied in much detail in nonlinear optics (see Agrawal and Boyd[1], Mills[43], Newell and Moloney [44], Bony, Metivier, Rauch [32]). Importance of nonlinear wave interactions for rotating and stratified fluids also is well known in physical literature. See, for example, Bartello [16], Bretherton [17], Farge and Sadourny [23], Lelong and Riley [35]), Müller *et al.* [42] , Warn [54]. In particular, they argue that the velocity field of a rotating, stably stratified fluid may be regarded as a superposition of waves which are modulated on the longer turbulence time scale.

Rigorous mathematical study of nonlinear hydrodynamic problems for rapidly rotating and stratified fluids with general initial data was started in Babin, Mahalov, Nicolaenko (henceforth BMN) [9,12,10], Embid and Majda [22], Grenier [29], Mahalov and Marcus [41]. Results of Grenier [29], Embid and Majda [22] are based on methods of Schochet [50]. In most detail the structure of the interactions, the nature of convergence and properties of the limit equations were studied by BMN [9,12]. In particular, for Euler and Navier-Stokes equations in a rapidly rotating frame with periodic boundary conditions they obtained the following results:

The general 3D problem was split into the 2D problem and into an infinite sequence of decoupled finite-dimensional ODE;

Analytic formulas which explicitly describe the dynamics of rapidly rotating fluids with general symmetric initial velocity fields were obtained;

Global regularity of solutions of the Navier-Stokes and Euler equations was proven;

Contribution of quasi-resonant terms was explicitly estimated and uniform explicit estimates of the error were obtained.

In BMN [14] the detailed structure of the limit Boussinesq equations was used to reveal the mechanism of frontogenesis in stably stratified fluids triggered by a weak rotation.

Approach of BMN is based on a detailed study of algebraic-geometric properties of dispersion relations and corresponding resonant conditions as well as on a detailed study of the structure of interactions.

These results strongly depend on geometry of the domain; one has to take into account whether the aspect ratios of the periodic box are singular (resonant) or non-resonant; it was shown in BMN [9,12] that the limit equations depend discontinuously on aspect ratios at singular points and it was proven in BMN [12] that it is impossible to get uniform, explicit estimates without restrictions on the aspect ratios (generic aspect ratios are non-resonant).

Similar results were obtained in BMN [10,14,15] for stratified fluids and in BMN [11,13] for shallow water equations.

Results of BMN [9,12,10] on the structure of nonlinear interactions were later used by Gallagher [24-26] in her study of smoothness of solutions of Navier-Stokes and Boussinesq equations.

The aim of the proposal is to study the impact of the boundary conditions on the nonlinear wave interactions and through this on such properties of equations as regularity of their solutions. We intend to study hydrodynamic equations in different types of domains. In particular, we consider periodic boxes with resonant aspect ratios. Resonant aspect ratios are interesting because the limit equations include non-trivial 3-wave interactions and there is a strong hope that classical 3-wave equations generically emerge as the limit subsystems of ODE. The study of effects of shallowness of the box is also of interest since in geophysical applications the vertical scale is usually much smaller than horizontal scale. Though results of BMN show that the thin layer case is in some respects better even when viscosity is zero, the combination of effects of thinness with fast rotation and viscosity was not studied in detail in previous research of BMN and is to be addressed in the proposed research.

We also intend to consider non-periodic boundary conditions, such as unbounded domain and cylindrical domain. These types of boundary conditions are especially interesting because of the following reasons:

First, they are rotational invariant (in contrast to periodic boundary conditions);

The second reason is that all the results (BMN [10,12], Gallagher [24-26], Avrin, Babin, Mahalov, Nicolaenko [5]) on global regularity of solutions strongly depend on the aspect ratios of the periodic box; therefore it is interesting to consider domains where the ratio does not appear.

The third reason is that in the case of rotation invariant domains the problem in a rotating frame is equivalent (after a change of variables) to the problem in non-rotating frame with the initial velocity field including large uniformly rotating component. Therefore positive results in this direction may help in solving the outstanding problem of global regularity of solutions of the Navier-Stokes system.

One more physical motivation for investigating the unbounded domain case is that nonlinear inertial wave effects which are crucial in the rapidly rotating case have a different nature in unbounded space: on one side, there is a wave propagation to infinity which should lessen nonlinear wave interactions, and resonances generated by multiple reflections in sides of the periodic box are absent; on the other side it is easier to excite resonant interactions thanks to the continuity of the spectrum.

Now we describe the problems under consideration in more detail.

We consider flows uniformly rotating with a constant angular velocity $\Omega_0/2$ about the x_3 axis. Let L_0 be the characteristic horizontal length scale and U_0 be the characteristic velocity. The problem is made dimensionless by referring lengths to L_0, velocities to U_0 and time to L_0/U_0, which leads to the introduction of a dimensionless rotation rate $\Omega/2$ (defined by $\Omega = \frac{\Omega_0 L_0}{U_0}$); Ω is assumed to be large and the Rossby number $1/\Omega$ is small.

In a frame of reference rotating with a constant angular velocity $\Omega/2$ about the x_3 axis, the Navier-Stokes equations (or the inviscid 3D Euler equations when $\nu = 0$) in the dimensionless variables have the form

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + \Omega \mathbf{J} \mathbf{U} - \nu \Delta \mathbf{U} = -\nabla p + \mathbf{F}, \quad \nabla \cdot \mathbf{U} = 0. \tag{0.1}$$

Here $\mathbf{U} = (U^1, U^2, U^3)$ is the velocity field, $\mathbf{J}$ is the rotation matrix, $\Omega\mathbf{J}\mathbf{U}$=$\Omega e_3 \times \mathbf{U}$ is the Coriolis force term, $\mathbf{F}$ is a force, we always assume $\nabla \cdot \mathbf{F} = 0$, and p is a modified pressure. We consider regular solutions of Eq. (0.1) $2\pi a_j$-periodic in every x_j taken along Cartesian axes e_j, $j = 1,2,3$; $a_1 = 1, 0 < a_2 \leq a_{21} \leq 1, 0 < a_3 \leq a_{31} \leq 1$. In the case $\nu = 0$ $\mathbf{F} = 0$.

The linear version of 0.1 was extensively studied by Poincaré in [46] . Poincaré's inertial waves are well-known in the geophysical literature (e.g. [27]). The mathematical theory of the Poincaré problem (0.2) has attracted a considerable amount of attention starting from the work of Sobolev [48]. The Poincaré propagator is the unitary group solution $\mathbf{E}(-\Omega t)\mathbf{\Phi}(0) = \mathbf{\Phi}(t)$ to the linear Poincaré problem:

$$\partial_t \mathbf{\Phi} + \Omega \mathbf{J} \mathbf{\Phi} = -\nabla p, \ \nabla \cdot \mathbf{\Phi} = 0. \tag{0.2}$$

We use Fourier series expansions for velocity fields $\mathbf{U}(x) = (U^1(x), U^2(x), U^3(x))$

$$\mathbf{U}(x) = \sum_n \exp(i(n_1 x_1 + n_2 x_2/a_2 + n_3 x_3/a_3))\mathbf{U}_n = \sum_n \exp(i\check{n} \cdot x)\mathbf{U}_n \tag{0.3}$$

where $\mathbf{U}_n$ are the Fourier coefficients, $[n_1, n_2, n_3] \in \mathbf{Z}^3$, $\check{n} = [n_1, n_2/a_2, n_3/a_3]$ are wavenumbers. We assume that all functions have zero average over the periodic parallelepiped.

Poincaré's [46] solutions can be written in Fourier space as:

$$\mathbf{E}_n(\Omega t) = \{\mathbf{E}(\Omega t)\}_n = \cos(\xi_n \Omega t)\mathbf{I} + \frac{1}{|\check{n}|}\sin(\xi_n \Omega t)\mathbf{R}_n, \ \xi_n = \frac{\check{n}_3}{|\check{n}|} = \frac{n_3}{a_3|\check{n}|} \tag{0.4}$$

where the matrix $i\mathbf{R}_n$ is the Fourier transform of the *curl* operator; we denote $|\check{n}|^2 = n_1^2 + n_2^2/a_2^2 + n_3^2/a_3^2$. The resonant interactions of Poincaré waves are present when the Poincaré frequencies satisfy the relation

$$\pm\xi_k \pm \xi_m \pm \xi_n = 0, \ k + m = n; \tag{0.5}$$

the corresponding set K of wavenumbers k, m, n is called the resonant set; when $\xi_k \xi_m \xi_n \neq 0$ the interactions are called 3-wave. We denote the set of resonant wavenumbers which do not correspond to 3-wave interactions by $\tilde{K}, \tilde{K} = K \cap \{k_3 m_3 n_3 = 0\}$ The set $\tilde{K}$ does not depend on a_3; we denote by $K^* = K \setminus \tilde{K}$ the 3-wave resonant wavenumber set, $K^* = K^*(\theta_2, \theta_3)$ strongly depends on the aspect ratios a_2, a_3. Important observation (see Section 3) is that the 3-wave resonant wavenumber set $K^* = K \setminus \tilde{K} = \emptyset$ for all $1/a_3^2 = \theta_3 \notin \Sigma_3^*(\theta_2)$ where $\Sigma_3^*(\theta_2)$ is a countable set.

As usual we denote x_3-averaging of a function U by $\overline{U}$, and for vector fields $\overline{\mathbf{U}}(x_1, x_2) = \frac{1}{2\pi a_3}\int_0^{2\pi a_3} \mathbf{U}(x_1, x_2, x_3)dx_3$. It is proven in BMN [9,12] that solutions of 3D Euler/Navier-Stokes equations can be decomposed as

$$\mathbf{U}(t, x_1, x_2, x_3) = \tilde{\mathbf{U}}(t, x_1, x_2) + \mathbf{V}(t, x_1, x_2, x_3) + \mathbf{r}. \tag{0.6}$$

Here $\tilde{\mathbf{U}}(t)$ approximates the vertical average of the exact solution $\overline{\mathbf{U}}(t)$, $\mathbf{V}(t)$ approximates $\mathbf{U} - \overline{\mathbf{U}}$ with $\overline{\mathbf{V}} = 0$; $\mathbf{r}$ is a remainder, its Sobolev norm is estimated from above by a quantity proportional to $1/\Omega$, for smooth enough initial data.

The field $\tilde{\mathbf{U}}(t)$ is a solution of the 2D Euler/Navier-Stokes system with vertically averaged initial data and force;

$$\partial_t\tilde{\mathbf{U}} + \tilde{\mathbf{U}}\cdot\nabla'\tilde{\mathbf{U}} - \nu\Delta'\tilde{\mathbf{U}} = \bar{\mathbf{F}} - \nabla'\tilde{p},\ \nabla\cdot\tilde{\mathbf{U}} = 0,\ \tilde{\mathbf{U}}(0) = \overline{\mathbf{U}}(0), \tag{0.7}$$

where $\nabla' = (\partial_1, \partial_2, 0)$, $\tilde{\mathbf{U}}\cdot\nabla'$ is the classical 2D Euler advection operator, $\nu \geq 0$; $\Delta' = \partial_1^2 + \partial_2^2$ is the 2D Laplace operator; if $\nu = 0$ it is the classical 2D Euler system. The system has 3-components and includes the equation for $\tilde{U}^3$.

The vector field $\mathbf{V}(t)$ which approximates $\mathbf{U} - \overline{\mathbf{U}}$ is related to the Poincaré velocity $\mathbf{v}(t)$ (the "amplitude" of Poincaré waves) by a Van der Pol type (see [30]) transformation $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$, which preserves all Sobolev norms. After the substitution the equations for $\mathbf{v}$ are Euler-like, but with time-dependent coefficients. The extended Euler/Navier-Stokes equations for $\mathbf{v}$ are for the velocity fields which are three-dimensional and three-component.

The remarkable property of the extended system for $\mathbf{v}$ is that (for generic, non-resonant a_2, a_3) *it splits into an infinite sequence of independent (uncoupled) subsystems of linear 8-component ordinary differential equations (ODE's)* for the corresponding (divergence-free) Fourier modes. Corresponding interactions are catalytic in terminology of Bartello [16] because they are activated only when the 2D field $\tilde{\mathbf{U}}(t)$ is non-zero. For a given index (n_1, n_2, n_3), $n_3 \neq 0$ they describe interactions of the quadruplet $\mathbf{U}_{(\pm n_1, \pm n_2, n_3)}$ of the Fourier modes. Each subsystem couples 8 non-autonomous ODE's (when $n_1 n_2 = 0$ the subsystem reduces to 4 or 2 equations) which are explicitly written (see [12]). The coefficients of this system are determined by the solution $\tilde{\mathbf{U}}(t, x_1, x_2)$ of 2D Euler equations (0.7)).

Such structure implies existence of infinite number of conservation laws for the extended Euler equations which is in contrast to the classical 3D Euler equations, where only energy and helicity are conserve, these conserved quantities of the extended Euler equations are the approximate adiabatic invariants ([2]) of 3D Euler equations in the small Rossby number situation. If the vertically averaged initial data are reflectional symmetric (that is satisfy zero flux boundary conditions in the half-sized box), the extended system is completely integrated and the exact solutions for $\mathbf{v}(t)$ and $\mathbf{V}(t)$ are obtained (see BMN [12]).
The equation for the "amplitude" $\mathbf{w}$ in the case of general (resonant or non-resonant) ratios a_2, a_3 was found and described in detail in BMN [8]. Nonuniform convergence to solutions of the limit equations was proven by Grenier [29] for every fixed a_2, a_3 for $\nu > 0$; one also can prove the non-uniform convergence using the approach of Schochet [50]. In BMN [9,12] it was proven that the solution of the limit equations

splits in the following way: $\mathbf{w} = \tilde{\mathbf{u}} + \mathbf{v}$. Here $\tilde{\mathbf{u}}$ is a solution of 2D Navier-Stokes equations as above. The vector-field $\mathbf{v}$ is a solution of the equations

$$\partial_t \mathbf{v} = -\nu \mathbf{A}\mathbf{v} + \mathbf{B}_{ex}(\tilde{\mathbf{u}}, \mathbf{v}) + \mathbf{B}_N^0(\mathbf{v}, \mathbf{v}) \tag{0.8}$$

with the initial condition $\mathbf{v}(0) = \mathbf{U}(0) - \bar{\mathbf{U}}(0)$. The operator $\mathbf{B}_{ex}(\tilde{\mathbf{u}}, \mathbf{v})$ continuously depends on a_3 (through periodic boundary conditions only) and is linear in x_3-dependent component $\mathbf{v}$. The bilinear operator $\mathbf{B}_N^0(\mathbf{v}, \mathbf{v})$ describes nonlinear interactions between 3 "fast" Poincaré waves; in Fourier representation it is given by

$$\mathbf{B}_N^0(\mathbf{v}, \mathbf{v})_n = \sum_{(k,m)\in K^*, k+m=n} Q_{kmn}(\check{k}, \check{m}, \mathbf{v}_k, \mathbf{v}_m). \tag{0.9}$$

The 3-component bilinear forms $Q_{kmn}(\check{k}, \check{m}, \mathbf{v}_k, \mathbf{v}_m)$ are given in explicit form in BMN [8], the only dependence of the forms on a_2, a_3 is through $\check{k}, \check{m}$, that is through boundary conditions. The domain of summation $K^* = K^*(\theta_2, \theta_3)$ depends on $(1/a_2^2, 1/a_3^2) = (\theta_2, \theta_3)$. For every fixed $1/a_2^2$ the summation set $K^*(\theta_2, \theta_3) \subset \mathbf{Z}^6$ is not empty only when $\theta_3 \in \Theta_3^*(\theta_2)$, the set $\Theta_3^*(\theta_2)$ is very thin, namely it is countable, we call it strict 3-wave resonant set. By now global regularity of solutions is not proven for solutions of the limit equations with resonant a_2, a_3; only local regularity like for 3D Euler equations is proven. In BMN [9,12], Gallagher [24-26] is studied the typical (and nice) case when 3-wave interactions are absent, that is $\theta_3 \notin \Theta_3^*(\theta_2)$ and $\mathbf{B}_N^0(\mathbf{v}, \mathbf{v})$ is identically zero. When $\theta_3 \in \Theta_3^*(\theta_2)$ is strictly resonant, $\mathbf{B}_N^0(\mathbf{v}, \mathbf{v})$ is non-zero and strongly depends on θ_3; the sets $K^*(\theta_2, \theta_3)$ with different resonant θ_3 do not intersect. It implies that the operator $\mathbf{B}_N^0$ depends on resonant θ_3 discontinuously, every point $\theta_3 \in \Theta_3^*$ is a point of discontinuity of the operator. Since $\mathbf{B}_N^0(\mathbf{v}, \mathbf{v})$ is not zero, *solutions of the limit system with general initial data discontinuously depend on* θ_3 as well (see [12]) and the convergence to solutions of the limit equations *cannot be uniform.* That is the intrinsic reason why one can not give explicit error estimates if the dependence on θ is ignored (like in [22], [29] , [50]); explicit, uniform estimates must include information on the θ parameters, see Theorem 1 below.

The main uniform convergence result of [12] shows that the convergence is uniform and the error is of order $(1/\mu_2 + 1/\mu_3)/\Omega$ when $\theta_2 \notin \Theta_2^\mu$, $\theta_3 \notin \Theta_3^\mu(\theta_2)$, with the Lebesgue measure $meas(\Theta_3^{\mu_3}) \leq \mu_3$, $meas(\Theta_2^{\mu_2}) \leq \mu_2$ with μ_2, μ_3 arbitrary small. This kind of convergence occurs in situations when the limit is a discontinuous function of a parameter θ when a small parameter $1/\Omega$ tends to zero non-uniformly; but usually it is possible to obtain explicit estimates only in very simple problems.

Theorem 1. *Let* $0 \leq \nu \leq 1$, $\theta_3 \notin \Theta_3^{\mu_3}(\theta_2)$, $\theta_2 \notin \Theta_2^{\mu_2}$, *let* $\sigma - \alpha > 6, \alpha > 3/2, M_{0\sigma} > 0$, $\mu_2, \mu_3 \leq 1$. *Let* $\|\mathbf{U}(0)\|_\sigma \leq M_{0\sigma}$, *let* $\|\mathbf{F}\|_\sigma + \|\partial_t \mathbf{F}\|_{\alpha+1} \leq M_{0\sigma}$ *for all* $t \geq 0$. *Let* $\mathbf{U}(t)$ *be an exact solution of 3D Euler equations (0.1) with* $\mathbf{U}(0) = \mathbf{U}_0(x_1, x_2, x_3)$. *Let* $\mathbf{E}(\Omega t)$ *be the Poincaré propagator,* $\mathbf{v}(t)$ *be the solution of the extended Euler system with initial data* $\mathbf{v}(0) = \mathbf{U}(0) - \bar{\mathbf{U}}(0)$. *Then the difference* $\mathbf{U}(t) - \tilde{\mathbf{U}}(t) - \mathbf{E}(-\Omega t)\mathbf{v}(t)$ *satisfies the estimate*

$$||\mathbf{U}(t) - \tilde{\mathbf{U}}(t) - \mathbf{E}(-\Omega t)\mathbf{v}(t)||_\alpha \leq C(1/\mu_2 + 1/\mu_3)/\Omega \text{ for } 0 \leq t \leq T_1 \tag{0.10}$$

where T_1 *depends only on* $M_{0\sigma}$, C *depends only on* $M_{0\sigma}, \alpha$.
Note that since $1/a_3^2 = \theta_3 \geq 1$, the Lebesgue measure of the almost resonant set $A_3^{\mu_3}$

of a_3 is related with μ_3 by the relation $\int_{A_3^{\mu_3}} 2a_3^{-3} da_3 = \mu_3$. Clearly for $0 < a_3 < a_{31}$ we obtain $meas(A_3^{\mu_3} \cap \{0 < a_3 < a_{31}\}) \leq \mu_3 a_{31}^3/2$. In typical in geophysical applications case a_{31} of order 0.01—0.1 the 3-wave almost resonant sets corresponding to $\mu_3 = 1$ for such domains are very sparse, which makes the case of the thin domain very special; this case is to be investigated in the proposed research.

Having estimates of convergence, one can easily derive regularity of solutions with large Ω from regularity of the limit equations. Therefore the main problem is regularity of solutions $\tilde{\mathbf{U}}(t) + \mathbf{v}(t)$ of the limit equations.

Existence of global strong solutions $\tilde{\mathbf{U}}(t)$ of 2D Euler equations is well-known (see, for example, Ebin and Marsden [21], Temam [53]). For quasigeostrophic equations (which play the role of 2D Euler in the case of stratified fluids) global regularity is proved by Bourgeois and Beale [18].

For non-resonant a_2, a_3 the regularity of solutions $\mathbf{v}(t)$ is proven in [12]; the case of resonant a_2, a_3 is to be studied in the proposed research.

Regularity of solutions of 3D Euler equations with rotation is proven in [9,12] on the interval $[0, T^*]$ where $T^* \to \infty$ as $\Omega \to \infty$. Regularity of solutions of Navier-Stokes equations (when the viscosity $\nu > 0$) is proven for all $t \geq 0$. These results are obtained under smoothness conditions on initial data $\mathbf{U}(0) \in H_\alpha$ and on the right-hand side and $\mathbf{F}$. Originally the smoothness requirements were very high, for example $\alpha \geq 42$ in [9]; in [12] it was assumed that $\alpha \geq 1$ for $\nu > 0$. Later the conditions on $\alpha, \mathbf{F}$ were relaxed by Gallagher [24-26]. The conditions on $\mathbf{U}(0), \mathbf{F}$ were further relaxed by Avrin, Babin, Mahalov, Nicolaenko [5] and now are natural; for example, in the viscous case it is sufficient for regularity that $\mathbf{U}(0) \in H_{1/2+\epsilon}$ and right-hand side satisfy

$$\int_T^{T+1} ||\mathbf{F}||^2_{-1/2+\epsilon} dt \leq M_F \quad \text{for all } T \geq 0. \tag{0.11}$$

Therefore the main restriction now is the restriction on a_2, a_3 which we will attack in the proposed research. The question of smoothness is closely related with the impact of 3-wave interactions on the energy propagation between Fourier modes which is an important question in rotating turbulence theory, and this question is non-trivial when aspect ratios are resonant.

The next theorem of [12] with improvements from [5] describes the situation when Ω is fixed and initial data are non-smooth and arbitrary large in H_0.

Theorem 2. *Let $\nu > 0, \theta_3 \notin \Theta_3^*(\theta_2)$. Let $||\mathbf{U}(0)||_0 \leq \rho_0$. Let $T^* = \rho_0^2/\nu$; there exists a number Ω' which depends only on M_F, ν, a_3, a_2 such that for every $\Omega \geq \Omega'$ and for any weak solution $\mathbf{U}(t)$ of 3DNS equations (0.1) which is defined on interval $[0, T^*]$ and satisfies classical energy inequality on this interval the following is true: $\mathbf{U}(t)$ can be extended to $0 < t < \infty$ and is regular for $T^* \leq t < +\infty$, it belongs to $H_{1/2+\epsilon}$ and $||\mathbf{U}(t)||_{1/2+\epsilon} \leq C_1(M_F, \nu)$ for every $t \geq T^*$. If $\mathbf{F}$ is time - independent, there exists a global attractor of the Navier-Stokes equation which has a finite fractal dimension and every such weak solution of the Navier-Stokes system is attracted to the global attractor as $t \to +\infty$.*

Note that when a_3, a_2 are fixed, the number Ω' does not depend on initial data at all, it depends only on the forcing $\mathbf{F}$ which can be arbitrary large.

Theorem 2 is in fact a generalization of the classical result of Leray on final regularization of solutions of 3D Navier-Stokes equations with zero forcing $\mathbf{F} = 0$ to the case of large arbitrary $\mathbf{F}$ compensated by fast rotation, see also [19], [20], [47].

As we mentioned, if the vertically averaged initial data $\bar{\mathbf{U}}(0)$ are reflectionally symmetric, explicit formulas express Fourier coefficients of solutions $v(t)$ of the extended Euler equations (0.8) in terms of initial data and time integrals of corresponding Fourier coefficients of $\tilde{\mathbf{U}}(t)$ (see [12] for details). This is possible because the equations for the Fourier coefficients are linear, have low dimension and special symmetry properties.

1. Euler and Navier-Stokes equations with periodic boundary conditions (the case of resonant aspect ratios)

When aspect ratios are resonant, the extended Euler equations (0.8) are not linear anymore. Nevertheless, the algebraic-geometric analysis made in BMN [12] can be applied to this situation as well. Since frequencies ξ_n of inertial Poincaré waves explicitly depend on aspect ratios a_2, a_3 (or on $\theta_2 = 1/a_2^2, \theta_3 = 1/a_3^2$), the resonant pairs θ_2, θ_3 form a countable union of algebraic curves in (θ_2, θ_3)-plane. Analysis which extends that of BMN [12] allows to build a hierarchy of nonlinear interactions and to classify resonant interactions which appear in the non-linear operator $\mathbf{B}_N^0(\mathbf{v}, \mathbf{v})$ in (0.9) using algebraic-geometric information concerning the equation (0.5).

Generically, arising nonlinear equations split into independent subsystems, the majority of which can be reduced (when $\nu = 0$) to classical integrable 3-wave systems of the type

$$\begin{aligned} \partial_t b_n &= c^1_{kmn} b_k b_m^*, \\ \partial_t b_m &= c^2_{kmn} b_k b_n^*, \\ \partial_t b_k &= c^3_{kmn} b_n b_m \end{aligned} \tag{1.1}$$

with $c^1_{kmn} + c^2_{kmn} + {c^3_{kmn}}^* = 0$. One of aims of the proposed research is to give the proof of such splitting.

That would give a mathematically rigorous derivation of the 3-wave equations as limit equations for wave interactions in rapidly rotating fluids; usual physical derivation uses a forceful truncation. The second application is global regularity of solutions of the extended Euler equations (0.8) and hence long-time existence of solutions for original fast-rotating Euler equations (similarly global regularity for the Navier-Stokes system when $\nu > 0$) for the case of resonant aspect ratios.

2. Rotating thin layers

The case of a rotating thin layer of fluid is of a special interest in geophysics. We shall apply the techniques of Babin, Mahalov, Nicolaenko to Navier-Stokes equations in a situation where the rotating layer is thin. In BMN the depth of the layer was fixed. Now we separately consider the case of a thin layer. In this situation it is possible to combine the techniques of BMN [12], Raugel and Sell [47] and Avrin [4] to find optimal conditions for regularity in the situation of rapidly rotating thin layers. The conditions under which solutions of the 3D Navier-Stokes system are regular for all times should include viscosity ν, thinness a_3 and the rate of rotation Ω and are expected to be of the form $a_3^{q_1}\Omega^{-q_2}\nu^{-q_3} \leq C$ where C depends on appropriate norms of initial data and right-hand side. The exponents q_1, q_3, q_2 and the appropriate smoothness conditions on initial data and right-hand sides have to be found in the proposed research.

3. Euler and Navier-Stokes equations with rapidly rotating initial fields in the unbounded domain

Two cases can be considered separately : the case of the 3D space and the case of the layer $0 < x_3 < \pi a_3$. In the second case either periodic or zero flux boundary condition can be imposed.

The algebra of the nonlinear interactions in both cases is the same as in the case of periodic boundary conditions studied in [12], the only (but essential) difference is that now the wavevectors k, m, n are not integer-valued but vary continuously. That creates a new mathematical difficulty, which has to be solved in the proposed research.

It is well-known that equations ([12]) with the Coriolis term are obtained through rewriting classical Navier-Stokes equations in a rotating coordinate system. Hence, the periodic boundary conditions correspond to rotating periodic boundary conditions in the original coordinate system. (This is a well-known model for Earth rotation starting from Rossby). The mathematical results of BMN [12] can be explained from a physical point of view in the following way. For typical values of aspect ratios the inertial waves interact weakly, do not build up resonances and this is reflected in possibility to prove global regularity of solutions. But for special values of aspect ratios the rotating boundary conditions create resonantly interacting waves. Note that periodic functions have infinite energy when considered in the unbounded space, therefore the impact of resonant interactions is very strong.

When one considers velocity fields with finite energy in the whole space, there are two new competing mechanisms influencing the nonlinear wave interaction. First, because the wavevectors are continuous, it is easier to build the resonant interactions. Second, since the space is unbounded and the total energy is finite (modulo rigid body rotation), the inertial waves transport energy to infinity thus weakening the interactions. More detailed mathematical analysis shows that the second mechanism is stronger if initial velocity fields decay fast enough. From a different point of view, the equation (0.5) determines the resonant surface in the space of wavenumbers k, m. Since denominators vanish on this surface, to study behavior of solutions one has to use the smoothness of the Fourier transform of the solution, and the smoothness is equivalent to a decay of the solution as $|x| \to \infty$.

Therefore one of essential parts of the proposed research is the detailed study of behavior at infinity, which can be done through estimates in weighted Sobolev spaces; one have to apply techniques similar to developed in the paper of proposer [6]. Using the geometry of the resonant surface and structure of resonant interactions the proposer can obtain the following type of theorem on global regularity of solutions of the Navier-Stokes system (0.1): Let $(1+|x|^2)^q\mathbf{U}_0 \in H_1$. Let Ω be large enough. Let $\mathbf{U}(0)=\mathbf{U}_0$. Then there exists a regular solution $\mathbf{U}(x,t)$ such that $\mathbf{U}(\cdot,t) \in H_1 \forall t \geq 0$.

After a change of variables x using a rotating coordinate system the equations for velocity fields written in the new system take the form of classical Navier-Stokes system without the Coriolis term:

$$\partial_t \mathbf{U}' + \mathbf{U}' \cdot \nabla \mathbf{U}' - \nu \Delta \mathbf{U}' = -\nabla p' + \mathbf{F}', \quad \nabla \cdot \mathbf{U}' = 0. \tag{3.1}$$

The initial data are transformed according to the relation $\mathbf{U}'(x,0) = \mathbf{U}(x,0) + \Omega e_3 \times x$. Therefore regularity of solutions of (0.1) implies regularity of solutions of (3.1) with corresponding initial data (for rotation invariant $\mathbf{F} = \mathbf{F}'$,for example $\mathbf{F} = \mathbf{0}$). So one readily obtains from the above statement concerning equation (0.1) the following theorem on global regularity of solutions for the classical Navier-Stokes system (3.1) without Coriolis force: Let $(1+|x|^2)^q\mathbf{U}'_0 \in H_1$. Let Ω be large enough. Let $\mathbf{U}'(0) = \mathbf{U}'_0 + \Omega e_3 \times x$. Then there exists a regular solution $\mathbf{U}'(t)$ such that $\mathbf{U}'(t) - \Omega e_3 \times x \in H_1 \forall t \geq 0$.

The initial data $\mathbf{U}'(0)$ here are not small, in fact they are very large.

Global regularity of solutions of the 3D Navier-Stokes system is the outstanding problem of applied analysis. Many studies are devoted to approach this problem (see Leray[37], Caffarelli, Kohn, Nirenberg[19], Constantin, Foias[20], Raugel and Sell [47], Scheffer[49], Temam[51] among many others). Generally speaking, global regularity is proven for the case of small or quasi-2D initial data, in general case only partial regularity is proven. The expected result gives regularity for arbitrary large initial data with a very large curl, it would be an important result by itself. It is well known that the lack of global bounds of the curl is the main obstacle to the proof of regularity. The proposed research can give a new tool to attack this problem, since, as results of BMN for rotating periodic boundary conditions show and we expect to obtain in the case of rotation-symmetric boundary conditions, the large curl helps to preserve smoothness rather than destroy it. Of course the success of the proposed research does not solve the classical problem , but it can be a major step on the way to solving it. According to Scheffer [49] using only energy estimate method one can get only the partial regularity result of Caffarelli, Kohn, Nirenberg[19], therefore new tools are required; and regularizing effects of curl-generated rotation may provide a new mechanism.

4. Euler equations for flows in a rotating circular cylinder.

The case of periodic boundary conditions for the Euler equations was considered in [12]. A similar method can be applied to the case of a circular cylinder, but the rigorous justification of all steps is much more difficult in this case and requires introducing new ideas. It is possible that only some of the results of [12 can be extended to this case (in a modified form). At the same time the case is important since the domain is rotation invariant.

The program of research is:

1. To study the structure of fast-time resonances

2. To derive the equations which describe the motion of a fluid in the limit $\Omega \to \infty$.

3. To obtain estimates of smoothness of solutions uniform in Ω on a finite interval.

3. To find conditions when the limit equation has regular solutions.

4. To find conditions when the limit equation splits into a sequence of decoupled finite-dimensional ODE.

5. To find conditions when the equations are integrable and to integrate them.

6. To find uniform estimates of the difference of solutions of the original and the limit equations

7. To prove long-time regularity of solutions of the original equations when the rate of rotation is high enough.