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Localization Phenomenon in Gaps of the Spectrum of Random Lattice Operators

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We consider a class of random lattice operators including Schrödinger operators of the form $H = -\Delta + w + gv$, where $w(x)$ is a real-valued periodic function, g is a positive constant and $v(x), x \in \mathbf{Z}^d$, are independent, identically distributed real random variables. We prove that if the operator $-\Delta + w$ has gaps in the spectrum and g is sufficiently small, then the operator H develops pure point spectrum with exponentially decaying eigenfunctions in a vicinity of the gaps.

Key words: random media, random potentials, Anderson model, Schrödinger operators, localization, gaps in the spectrum

Introduction.

We consider a matrix operator $H = H_0 + gv$ acting in $l^2(\mathbf{Z}^d)$ as follows

$$(H\psi)(x) = \sum_{y \in \mathbf{Z}^d} H_0(x,y)\psi(y) + gv(x)\psi(x), x \in \mathbf{Z}^d \tag{1}$$

where $v(x), x \in \mathbf{Z}^d$, are real, independent, identically distributed random variables, g is a positive constant and H_0 is a local periodic operator in the following sense: there exists a natural number ρ (called the range of H_0) such that if $|x-y| > \rho$ then $H_0(x,y) = 0$, and there exist a vector $q = (q_1, \ldots, q_d) \in \mathbf{Z}^d$ with positive components such that $H_0(x,y) = H_0(x+q', y+q'), \forall x, y \in \mathbf{Z}^d$ and $\forall q' \in q_1\mathbf{Z} \times \ldots \times q_d\mathbf{Z}$. We show that the spectrum of such an operator H_0 consists of a finite number of intervals which we shall

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call bands of the spectrum; the intervals between bands of the spectrum (if any) being the gaps in the spectrum. One can easily construct local periodic operators exhibiting gaps in the spectrum. For instance, let $H_0 = -\Delta + aw$, where Δ is the lattice Laplacian, a is a positive constant and w is the operator of the multiplication by a real, periodic, nonconstant function $w(x)$, so H_0 is a local periodic operator. Since Δ is a bounded operator, it is clear that H_0 has gaps in the spectrum if the constant a is large enough. Another example of a periodic operator H_0 exhibiting gaps in the spectrum is constructed in [1].

According to the philosophy of Anderson localization, localized states can appear in a vicinity of movable edges of gaps in the spectrum, i.e. such edges that depend on random coefficients [2,3]. It is known that operators of the form (1) with probability 1 have pure point spectrum with exponentially decaying eigenfunctions for low energies, i.e. far enough from the spectrum of H_0 [4-11], and also near the end points of the spectrum [15]. We prove here that if the spectrum of the operator H_0 has gaps, then for a sufficiently small constant g the random operator H with probability 1 develops pure point spectrum with exponentially decaying eigenfunctions in a vicinity of all gaps of the operator H_0.

Our proof of localization in the gaps is based on the multiscale method used by von Dreifus and Klein [9] and Spencer [15], and on the relevant spectral properties of periodic operators and their restrictions to finite domains that we develop in this paper.

1. Statement of Results.

We begin with a precise definition of a local periodic operator. Let D be a natural number and $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ be the Hilbert space of $\mathbf{C}^D-$ *valued* functions $\varphi(x)$, with the standard norm $\|\varphi\|^2 = \sum |\varphi(x)|^2$. Let us denote by $\mathcal{L}_D$ the linear space of all $\mathbf{C}^D-$ *valued* functions $\varphi(x)$. If $D = 1$ we shall just write $l^2(\mathbf{Z}^d)$ and $\mathcal{L}$ in place of $l^2(\mathbf{Z}^d, \mathbf{C}^1)$ and $\mathcal{L}_1$ respectively. Now we introduce a matrix H_0 with entries $H_0(x, y)$, $x, y \in \mathbf{Z}^d$, which are in turn $D \times D-$ *matrices* with complex entries . We shall consider here just symmetric matrices H_0, thus $H_0(x, y) = H_0^*(y, x), x, y \in \mathbf{Z}^d$, where for a matrix (operator) A the adjoint to its matrix (operator) is denoted by A^*. We define a norm $|x|_\infty$ for $x = (x_1, \ldots, x_d) \in \mathbf{Z}^d$ as follows

$$|x|_\infty = \max_{1 \le j \le d} |x_j|$$

Definition. *We shall call a matrix A local if there is a natural number ρ such that $A(x, y) = 0$, whenever $|y - x|_\infty > \rho$. For a vector $q = (q_1, \ldots, q_d) \in \mathbf{Z}^d$ with positive* *coordinates we shall call a matrix A $q-periodic$ (or just periodic) if it is local and the following equalities hold*

$$A(x,y) = A(x+q', y+q'), \forall x, y \in \mathbf{Z}^d, \forall q' \in q_1\mathbf{Z} \times \ldots \times q_d\mathbf{Z} \tag{1.1}$$

We associate with any periodic matrix H_0 an operator denoted by same symbol whose action is defined in standard fashion by $(H_0\psi)(x) = \sum_y H_0(x,y)\psi(y)$. Clearly, a periodic operator H_0 is correctly defined as an operator from $\mathcal{L}_D$ to $\mathcal{L}_D$ and it is a bounded self-adjoint operator in $l^2(\mathbf{Z}^d, \mathbf{C}^D)$. In particular, a $q-periodic$ operator H_0 maps any $q-periodic$ function ψ onto a $q-periodic$ function $H_0\psi$.

Remark. *If $H_0 = -\Delta + w$ where w is the operator of the multiplication by a $q-periodic$ function, then H_0 is a $q-periodic$ operator.*

Schrödinger operators with periodic potentials on $\mathbf{R}^d$ are the subject of the well known Floquet-Bloch theory [12]. Since modifications needed to extend the theory to the lattice case are hard to find in the literature, we will state and prove what we need.

Theorem 1. (band structure of spectrum). *If H_0 is a periodic operator on $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ then its spectrum σ_0 consists of a finite number J of intervals, namely*

$$\sigma_0 = \cup_{1\le i\le J}[\mu_i^{(0)}, \lambda_i^{(0)}];\ 0 \le \mu_i^{(0)} \le \lambda_i^{(0)}, 1 \le i \le J,\ \ \lambda_i^{(0)} < \mu_{i+1}^{(0)}, 1 \le i \le J-1 \tag{1.2}$$

Definition. (gaps). *We call the above intervals bands. If $J > 1$ then we shall call the intervals $(\lambda_i^{(0)}, \mu_{i+1}^{(0)})$, $1 \le i \le J-1$, gaps in the spectrum (or just gaps).*

We have already discussed in the introduction that periodic operator with gaps in the spectrum can be easily constructed, in particular, the lattice Schrödinger operator of the form $H_0 = -\Delta + w$ with a periodic potential may have gaps in the spectrum. Thus, we shall just assume the existence of gaps in the spectrum of the operator H_0.

¿From now on we always have $D = 1$, unless stated otherwise. The main operator we are interested in is the operator $H = H_0 + gv$ where g is a positive constant and the operators H_0 and v satisfy the following assumptions:

Assumption H. *H_0 is a $q-periodic$ self-adjoint operator on $l^2(\mathbf{Z}^d)$ with $J-1 > 0$ gaps $(\lambda_i^{(0)}, \mu_{i+1}^{(0)})$, $1 \le i \le J-1$.*

Assumption V. *v is the operator on $l^2(\mathbf{Z}^d)$ given by multiplication by $v(x)$, where $v(x), x \in \mathbf{Z}^d$, are independent, identically distributed random real-valued variables on a probability space with probability measure $\mathbf{P}$. The probability distribution μ of $v(0)$ has a* *bounded density* φ *with* $\|\varphi\|_\infty \le D_0$. *For convenience we take* $\mathcal{R}(v(x)) = [-1,1]$ *where* $\mathcal{R}(v(0))$ *is the essential range of the random variable* $v(0)$.

Theorem 2. (location of the spectrum). *Let* $\xi(x) = \xi_\omega(x), x \in \mathbf{Z}^d$, *be a set of real-valued independent, identically distributed random variables on the probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ $(\omega \in \Omega)$ *such that for some finite constants* ξ_1, ξ_2 *we have*

$$\mathcal{R}(\xi(x)) = [\xi_1, \xi_2] \tag{1.3}$$

Suppose that the operator H *acts in the Hilbert space* $l_2(\mathbf{Z}^d)$ *and* $H = H_0 + \xi$, *where* H_0 *satisfies Assumption H and* ξ *is the operator given by multiplication by the function* $\xi(\cdot)$. *Then the following statements hold:*
(i) with probability 1 *the spectrum* $\sigma(H)$ *of the operator* H *is nonrandom, i.e., there exists a closed set* $\sigma \subseteq \mathbf{R}$ *such that with probability* 1 $\sigma(H) = \sigma$; *in addition to that, with probability* 1 *the spectrum can be represented as follows*

$$\sigma(H) = \sigma = \sigma(H_0) + \mathcal{R}(\xi(x)) = \sigma(H_0) + [\xi_1, \xi_2] \tag{1.4}$$

where for two subsets $A, B \subseteq \mathbf{R}$ $\quad A + B = \{\lambda + \mu : \lambda \in A, \mu \in B\}$;
(ii) let us set $\xi(x) = gv(x)$ *where* v *satisfies Assumption V; if we use the notations of Theorem 1 and introduce* g_i *by the following equality*

$$g_i = (\mu_{i+1}^{(0)} - \lambda_i^{(0)})/2,\ 1 \le i \le J-1, \tag{1.5}$$

then for any $0 \le g < g_i$ *with probability* 1 *the spectrum* $\sigma(H) = \sigma$ *has a nonempty gap*

$$(\lambda_i, \mu_{i+1}),\ \lambda_i = \lambda_i^{(0)} + g < \mu_{i+1} = \mu_{i+1}^{(0)} - g \tag{1.6}$$

which is associated naturally with the gap $(\lambda_i^{(0)}, \mu_{i+1}^{(0)})$ *in the spectrum of the unperturbed periodic operator.*

In other words, Theorem 2 says that the spectrum of the random operator H is nonrandom and if the constant g is small enough then it has a band-gap structure associated naturally with the spectrum of the operator H_0. Moreover, taking the coefficient g small enough we can keep open up any gap in the spectrum of the unperturbed periodic operator.

The main statement of this paper is the following.

Theorem 3. *Let* $H = H_0 + gv$, *where* v *and* H_0 *satisfy Assumptions V and H,* *respectively. Assume also that for some* i, $1 \le i \le J-1$, *we have* $0 \le g < g_i$. *Let us pick numbers* $\Omega_\pm$ *such that* $-1 < \Omega_- < 0 < \Omega_+ < 1$. *Then there exists a positive number* $\widetilde{p}_+ = \widetilde{p}_+(d, H_0, D_0, \Omega_+, g)$ *(or respectively a positive number* $\widetilde{p}_- = \widetilde{p}_-(d, H_0, D_0, \Omega_-, g)$*) such that if* $p_+ = \mu\{[\Omega_+, 1]\} < \widetilde{p}_+$ $(p_- = \mu\{[-1, \Omega_-]\} < \widetilde{p}_-)$, *we can find* $\delta_+ > 0$ $(\delta_- > 0)$ *such that* H *is exponentially localized in the interval* $(\lambda_i - \delta_+, \lambda_i)$, $((\mu_i, \mu_i + \delta_-))$ *with probability* 1. *Moreover,*

$$\lim_{p_+ \to 0} \delta_+ = g(1 - \Omega_+) \tag{1.7}$$

with a similar statement being true for δ_-.

We also prove a somewhat different version of Theorem 3.

Theorem $3'$. *Let* $H = H_0 + gv$ *as in Theorem 3, and in addition suppose that* $\mu\{|v(0) \pm 1| \le \varepsilon\} \le C\varepsilon^\eta$, *for a finite constant* C *and a constant* $\eta > d$. *Then, if* $0 \le g \le g_i$ *we can find* $\delta_\pm(d, H_0, D_0, g, C, \eta)$ *such that* H *is exponentially localized in the interval* $(\lambda_i - \delta_+, \lambda_i)$, $((\mu_i, \mu_i + \delta_-))$ *with probability* 1.

The proofs of Theorems 2, 3 and $3'$ are based on auxiliary statements concerning the relationship of the spectrum of a periodic operator A and its periodic restrictions to finite parallelepipeds in $\mathbf{Z}^d$: they will be formulated as theorems below. In order to so, we introduce the following notations. If $u, v \in \mathbf{Z}^d$ then $uv = (u_1 v_1, \ldots, u_d v_d) \in \mathbf{Z}^d$.

Definition. *Let* $u, v \in \mathbf{N}^d$. *If* $v = nu$ *for some* $n \in \mathbf{N}^d$ *we will write* $u \preceq v$. *If in addition, all the coordinates of* n *are strictly greater than* 1, *we will write* $u \prec v$.

Definition. *For* $u \in \mathbf{N}^d$ *we define a parallelepiped* $C^u = \{0, \ldots, u_1 - 1\} \times \ldots \times \{0, \ldots, u_d - 1\} \subset \mathbf{Z}^d$. *We will write* $C_u \preceq C_v$, *or* $C_u \prec C_v$ *if* $u \preceq v$ *or* $u \prec v$, *respectively.*

Suppose now that A is a $q-$*periodic* self-adjoint operator in $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ and $u \succeq q$. Then we introduce a finite matrix $\overset{\circ}{A}_{C^u}$ associated with the operator A as follows. Let

$$\overset{\circ}{A}_{C^u}(x, y) = \sum_{n \in \mathbf{Z}^d} A(x, y + nu), \;\; x, y \in \mathbf{Z}^d \tag{1.8}$$

Now, we define $\overset{\circ}{A}_{C^u} = \{\overset{\circ}{A}_{C^u}(x, y), x, y \in C^u\}$. If $u = q$ will shall just write

$$\overset{\circ}{A} = \overset{\circ}{A}_{C_q} \tag{1.9}$$

We call the matrix $\overset{\circ}{A}_{C^u}$ the periodic restriction of the local operator A to the parallelepiped $C^u, u \succeq q$. Let us denote by $\sigma(A)$ the spectrum of an operator (or matrix) A.

Theorem 4. *Let A be a $q-periodic$ self-adjoint operator in $l^2(\mathbf{Z}^d,\mathbf{C}^D)$. Suppose that $C_n, n=1,2,\dots$ is a sequence of parallelepipeds such that $C^q \preceq C_n \prec C_{n+1}, n\geq 1$. Then*

$$\sigma(A) = \overline{\cup_{n\geq 1}\sigma(\overset{\circ}{A}_{C_n})}\ ,\ \ \sigma(\overset{\circ}{A}_{C_n}) \subseteq \sigma(\overset{\circ}{A}_{C_{n+1}}) \subseteq \sigma(A) \tag{1.10}$$

This theorem enables us to control the spectrum in vicinities of gaps of the periodic restrictions of the operator H to finite parallelepipeds.

2. Proof of Theorems 1, 2 and 4 .

In this section we investigate the location of the spectrum of the operators H and H_0. We need first to extend some aspects of the well known Floquet-Bloch theory to the periodic operators H_0 following the scheme developed for multidimensional periodic Schrödinger operators in [12].

Floquet-Bloch theory for lattice periodic operators

Let A be a $q-periodic$ self-adjoint operator in $\mathcal{L}_D$ with entries $A(x,y), x,y\in\mathbf{Z}^d$, defined in the previous section, and let $V_j, 1\leq j\leq d$ be the unitary shift operators acting on Hilbert spaces $l^2(\mathbf{Z}^d,\mathbf{C}^D)$ which act as follows. If $e_j, 1\leq j\leq d$, are the standard basis vectors in the lattice $\mathbf{Z}^d$ then V_j are defined by formulas

$$(V_j\Psi)(x) = \Psi(S_j(x)),\ S_j(x) = x - e_j, x\in\mathbf{Z}^d, \quad 1\leq j\leq d. \tag{2.1}$$

That is, S_j stands for the shift in the lattice $\mathbf{Z}^d$ by the vector e_j. To proceed further we need an appropriate description of $q-periodic$ operators. We adopt here the following notations:

M^D is the set of $D\times D-matrices$ with complex entries;
$\mathcal{F}_q^D$ is the set of $q-periodic$ $\mathbf{C}^D-valued$ functions $\Psi(x), x\in\mathbf{Z}^d$;
$\mathcal{M}_q^D$ is the set of $q-periodic$ $M^D-valued$ functions $a(x), x\in\mathbf{Z}^d$;
$\mathcal{A}_q^D$ is the set of $q-periodic$ operators.
$V^z = V_1^{z_1}\dots V_d^{z_d}$, $z\in\mathbf{Z}^d$;
if $a(\cdot)\in\mathcal{F}_q^D$ and $z\in\mathbf{Z}^d$ $a^{(z)}(x) = a(x-z), x\in\mathbf{Z}^d$.

Lemma 2.1 . *Let a be the operator given by multiplication by the periodic function $a(\cdot)\in\mathcal{M}_q^D$. Then*
(i) for any $a(\cdot)\in\mathcal{M}_q^D$ and $z\in\mathbf{Z}^d$ $a, V^z\in\mathcal{A}_q^D$;
(ii) A is a periodic operator with entries $A(x,y), x,y\in\mathbf{Z}^d$, i.e. $A\in\mathcal{A}_q^D$ if and only if

there exist a finite positive ρ and a collection of $q-periodic$ functions $a_z(\cdot) \in \mathcal{F}_q^D, z \in \mathbf{Z}^d$, and $|z| \leq \rho$ such that the following representation is true

$$A = \sum_{|z|\leq\rho} a_z V^z, a_z(x) = A(x, x-z), x \in \mathbf{Z}^d \tag{2.2}$$

If in addition A is a self-adjoint operator then the following equalities hold

$$a_z^*(x) = a_{-z}(x-z) = a_{-z}^{(z)}(x),\ x, z \in \mathbf{Z}^d, |z| \leq \rho \tag{2.3}$$

Moreover, $\mathcal{A}_q^D$ is an algebra and for any $a(\cdot) \in \mathcal{F}_q^D$ and $z \in \mathbf{Z}^d$ we have

$$aV^z = V^z a^{(-z)} \tag{2.4}$$

Proof. The proof follows immediately from the definition of a $q-periodic$ operator and operators V_j. □

For any parallelepiped $C^u, u \succeq q$, and a $q-periodic$ operator A we have defined the matrix $\overset{\circ}{A}_{C^u}$ by formula (1.8) and called it the periodic restriction of A to C^u. This periodic restriction possesses the following properties.

Lemma 2.2 . *Let A be a $q-periodic$ operator with entries $A(x,y), x, y \in \mathbf{Z}^d$, and $C^u, u \succeq q$. Then the function $\overset{\circ}{A}_{C^u}(x,y)$ defined by formula (1.8) for any $x, y \in \mathbf{Z}^d$ is $u-periodic$ with respect to both x and y. Namely*

$$\overset{\circ}{A}_{C^u}(x+nu, y) = \overset{\circ}{A}_{C^u}(x, y+nu) = \overset{\circ}{A}_{C^u}(x,y), x, y, n \in \mathbf{Z}^d \tag{2.5}$$

In addition, if A is a self-adjoint operator then the finite matrix $\overset{\circ}{A}_{C^u}(x,y),\ x, y \in C^u$ is also self-adjoint. If B is another $q-periodic$ operator then the following identity holds:

$$(\overset{\circ}{AB})_{C^u} = \overset{\circ}{A}_{C^u}\overset{\circ}{B}_{C^u} \tag{2.6}$$

Proof. The statements of the lemma easily follows from the definition of $q-periodic$ operators, in particular (1.1). □

It is clear from (2.4) that a $q-periodic$ A commutes with the operators $V_j^{q_j}, 1 \leq j \leq d$. Based on this fact, we shall introduce an operator $\widehat{A}$ which is on one hand unitarily equivalent to A, and on the other hand can be decomposed into fibers $\widehat{A}(\kappa)$ by the direct integral

$$\widehat{A} = \int_M^{\oplus} \widehat{A}(\kappa) d\kappa, M = [0, q_1^{-1}] \times \ldots \times [0, q_d^{-1}] \tag{2.7}$$

where $\widehat{A}(\kappa)$ is a $|Q| \times |Q| - matrix$ depending on κ. In order to do so, we consider the Fourier transform F for $\Psi \in l^2(\mathbf{Z}^d, \mathbf{C}^D)$ defined by the formulas

$$[F\Psi](k) = \widetilde{\Psi}(k) = \sum_{x \in \mathbf{Z}^d} e^{2\pi ikx} \Psi(x) \tag{2.8}$$

$$\Psi(x) = [F^{-1}\widetilde{\Psi}](x) = \int_K \widetilde{\Psi}(k) e^{-2\pi ikx} dk, K = [0, 1]^d \tag{2.9}$$

which is a unitary transform of $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ to $L^2(K, \mathbf{C}^D)$, i.e. the Hilbert space of $\mathbf{C}^D - valued$ functions on K which are square integrable with respect to Lebesgue measure dk. We shall also consider the Fourier transform of the operator A and denote it by $\widetilde{A} = FAF^{-1}$. As it follows from the previous formula $\widetilde{\Psi}(k)$ can be viewed as a $(1, \ldots, 1) - periodic$ function on $\mathbf{R}^d$.

Now, to use the $q-periodicity$ of the operator A and to handle $q-periodic$ functions on the lattice $\mathbf{Z}^d$ it is convenient to introduce the discrete torus

$$\mathbf{Q} = \mathbf{Q}_q = \mathbf{Z}^d / \mathbf{Z}_q^d, \mathbf{Z}_q^d = q_1 \mathbf{Z} \times \ldots \times q_d \mathbf{Z} \tag{2.10}$$

where $\mathbf{Z}^d$ is treated as a ring with the ordinary operation of addition and the following operation of multiplication for $a, b \in \mathbf{Z}^d : (ab)_j = a_j b_j, 1 \leq j \leq d$. Clearly, $\mathbf{Q}$ as a set can be identified naturally with the parallelepiped $Q = C^q$, and we will identify a $q - periodic$ complex-valued function on $\mathbf{Z}^d$ with the appropriate function on $\mathbf{Q}$ (or Q). The space of $\mathbf{C}^D - valued$ functions on $\mathbf{Q}$ will be denoted by $\mathbf{C}^{D, \mathbf{Q}}$. We introduce the scalar product for $\Phi, \Psi \in \mathbf{C}^{\mathbf{Q}}$ by

$$\Phi \cdot \Psi = \sum_{m \in \mathbf{Q}} \Phi_m^* \Psi_m \tag{2.11}$$

where Φ^* is the vector adjoint to Φ. We also introduce the Fourier transform $\breve{\Psi} = F_q \Psi$ of the $\mathbf{C}^D - valued$ functions Ψ on the discrete torus $\mathbf{Q}$ in the ordinary way by

$$\breve{\Psi}_l = [F_q \Psi]_l = |Q|^{-1/2} \sum_{m \in \mathbf{Q}} e^{2\pi iml/q} \Psi_m, l \in \mathbf{Q}, \quad F_q^* F_q = I \tag{2.12}$$

where I stands for the identity matrix and F_q^* is the matrix adjoint to F_q. In fact, F_q is a unitary matrix.

Returning to the construction of the direct integral (2.7) we decompose the parallelepiped K into equal smaller parallelepipeds as follows

$$K = \cup_{l \in Q} M_l, M_l = M + l/q, l = (l_1, \dots, l_d), q = (q_1, \dots, q_d) \in \mathbf{Z}^d$$

where (2.13)

$$l/q = (l_1/q_1, \dots, l_d/q_d)$$

and consider the corresponding decomposition of a function $\widetilde{\Psi} \in L^2(K, \mathbf{C}^D)$

$$\widetilde{\Psi} : \{\widetilde{\widetilde{\Psi}}_l(\kappa), \kappa \in M, l \in Q\}, \widetilde{\widetilde{\Psi}}_l(\kappa) = \widetilde{\Psi}(\kappa + l/q) \tag{2.14}$$

As it follows from this formula the function $\widetilde{\widetilde{\Psi}}_l(\kappa)$ is a $q - periodic$ function of $l \in \mathbf{Z}^d$. So, if we introduce $\widehat{\Psi}(\kappa) = \{\widetilde{\widetilde{\Psi}}_l(\kappa), l \in Q\}$ and the Hilbert space $L^2(M, \mathbf{C}^{D,\mathbf{Q}})$ (i.e., the Hilbert space of $\mathbf{C}^{D,\mathbf{Q}} - valued$ functions on M which are square integrable with respect to Lebesgue measure $d\kappa$), then based on the formula (2.14) one can define the unitary operator W

$$[W\widetilde{\Psi}](k) = \widehat{\Psi}(\kappa), \; W : L^2(K, \mathbf{C}^D) \mapsto L^2(M, \mathbf{C}^{D,\mathbf{Q}}) \tag{2.15}$$

Therefore, we have the following representation of $L^2(K, \mathbf{C}^D)$ by the constant fiber direct integral

$$WL^2(K, \mathbf{C}^D) = L^2(M, \mathbf{C}^{D,\mathbf{Q}}) = \int_M^{\oplus} \mathbf{C}^{D,\mathbf{Q}} d\kappa \tag{2.16}$$

For an operator A in $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ we shall denote $\widehat{A} = WFA(WF)^{-1}$. From the definitions (2.1) of the operators V_j we easily obtain

$$[\widehat{V}_j \widehat{\Psi}]_l(\kappa) = exp\{2\pi i(\kappa_j + l_j/q_j)\}\widehat{\Psi}_l(\kappa), 1 \le j \le d, l \in \mathbf{Q}, \kappa \in M \tag{2.17}$$

In order to find the appropriate representation for the operator A we use Lemma 2.1 and represent $q - periodic$ functions $a_z(x)$ as follows

$$a_z(x) = \sum_{l \in Q} \breve{a}_{z,l} exp\{-2\pi i(l/q)x\}, \; \breve{a}_{z,l+\alpha q} = \breve{a}_{z,l}, x, l, \alpha \in \mathbf{Z}^d \tag{2.18}$$

where

$$\breve{a}_{z,l} = |Q|^{1/2}[F_q a'_z]_l, , a'_z = [a_{z,m}, m \in \mathbf{Q}\}, \; a_{z,m} = a_z(m), m \in \mathbf{Q} \tag{2.19}$$

Then, taking into account (2.14) we get

$$[\widetilde{a_z\Psi}](k) = \sum_{m\in Q} \breve{a}_{z,m}\widetilde{\Psi}(k-m/q), \quad [\widehat{a_z\Psi}]_l(\kappa) = \sum_{m\in\mathbf{Q}} \breve{a}_{z,l-m}\widehat{\Psi}_m(\kappa), l\in\mathbf{Q} \quad (2.20)$$

For any operator (matrix) B acting in the finite-dimensional space $\mathbf{C}^{D,\mathbf{Q}}$ we shall denote by $\breve{B}$ the following operator (matrix)

$$\breve{B} = F_q B F_q^{-1} \quad (2.21)$$

Lemma 2.3 . *Let $U_j, 1\le j\le d$ be the unitary matrices on $\mathbf{C}^{D,\mathbf{Q}}$ defined by*

$$[U_j\Psi]_l = \Psi_{l-e_j}, l\in\mathbf{Q} \quad (2.22)$$

and hence

$$[\breve{U}_j\Psi]_l = exp\{2\pi i l_j/q_j\}\Psi_l, l\in\mathbf{Q} \quad (2.23)$$

Let b_l be a $\mathcal{M}^D$ – valued function on the torus $\mathbf{Q}$ and denote by b the operator given by multiplication by the function b_l in the finite dimensional space $\mathbf{C}^{D,\mathbf{Q}}$. Denote by $b_l^{(z)} = b_{l-z}, l\in\mathbf{Q}, z\in\mathbf{Z}^d$, where $l-z$ is understood modulo q. Then the following relationships hold

$$[\breve{b}\Psi]_l = \sum_{m\in\mathbf{Q}} \breve{b}_{l-m}\Psi_m, l\in\mathbf{Q} \quad (2.24)$$

$$bU^z = U^z b^{(-z)}, z\in\mathbf{Z}^d \quad (2.25)$$

Proof. The statement of the lemma follows immediately from (2.12) and (2.21). □

Lemma 2.4 . *Let $\breve{a}_z, |z|\le\rho$ be matrices on $\mathbf{C}^{D,\mathbf{Q}}$ defined by formulas*

$$[\breve{a}_z\Psi]_l = \sum_{m\in\mathbf{Q}} \breve{a}_{z,l-m}\Psi_m, l\in\mathbf{Q} \quad (2.26)$$

Then the following relationships are true

$$[\widehat{V_j\Psi}](\kappa) = e^{2\pi i\kappa_j}\breve{U}_j\widehat{\Psi}(\kappa), \quad [\widehat{A\Psi}](\kappa) = [\sum_{|z|\le\rho} \breve{a}_z e^{2\pi i(\kappa z)}\breve{U}^z\,]\widehat{\Psi}(\kappa), \kappa\in M \quad (2.27)$$

In addition to that, the operator A has the desired fiber structure (2.7) *and for the matrices $\widehat{A}(\kappa)$ the following representation is valid*

$$\widehat{A}(\kappa) = \sum_{|z|\le\rho} \breve{a}_z e^{2\pi i(\kappa z)}\breve{U}^z, \kappa\in M \quad (2.28)$$

The matrices $\widehat{A}(\kappa), k\in M$ are self-adjoint.

Proof. The proof of (2.27) follows straightforwardly from (2.2), (2.17), (2.19) and (2.20). In turn, the equality (2.28) is a consequence of (2.27) and (2.14) – (2.16). The self-adjointness of $\widehat{A}(\kappa)$ follows from (2.28), (2.27), (2.3), (2.25). □

Lemma 2.5 . *Let us introduce the following operators in* $l^2(\mathbf{Z}^d, \mathbf{C}^D)$

$$V_j(\kappa) = e^{2\pi i\kappa_j} V_j, 1 \le j \le d, \;\; A(\kappa) = \sum_{|z| \le \rho} a_z V(\kappa)^z \tag{2.29}$$

Then,

$$F_q^{-1} e^{2\pi i\kappa_j} U_j F_q = \overset{\circ}{V}_j(\kappa), \;\; F_q^{-1} \widehat{A}(\kappa) F_q = \overset{\circ}{A}(\kappa) \tag{2.30}$$

Proof. The statements of the lemma follows from (1.9) and Lemmas 2.2 and 2.4. □

Theorem 2.6 . *If* $\mathcal{U} = F_q^{-1} W F$ *and* A *is a* $q-$*periodic self-adjoint operator then we have*

$$\mathcal{U} A \mathcal{U}^{-1} = \int_M^{\oplus} \overset{\circ}{A}(\kappa) d\kappa, M = [0, q_1^{-1}] \times \ldots \times [0, q_d^{-1}] \tag{2.31}$$

where the direct integral decomposition acts in the Hilbert space

$$\int_M^{\oplus} \mathbf{C}^{D,\mathbf{Q}} d\kappa \tag{2.32}$$

In particular, the spectrum $\sigma(A)$ *can be represented in the form*

$$\sigma(A) = \cup_{\kappa \in M} \sigma(\overset{\circ}{A}(\kappa)) \tag{2.33}$$

Proof. The equality (2.31) follows immediately from Lemmas 2.4 and 2.5, whereas the representation (2.33) is a direct consequence of (2.31). □

Proof of Theorem 1. In view of the representation (2.33) the spectrum $\sigma(A)$ equals to the union of the range of values of the set of real functions $\lambda_l(\kappa), \kappa \in M (l \in \mathbf{Q})$ which are respectively the eigenvalues of the matrices $\overset{\circ}{A}(\kappa)$. It easily follows from Lemmas 2.4 and 2.5 that the matrices $\overset{\circ}{A}(\kappa)$, and therefore their eigenvalues, are continuous functions of κ. This means that the union of the sets described above must consist of a finite number of intervals. This completes the proof of Theorem 1. □

To prove Theorem 2 we will need some more auxiliary statements for the $q-periodic$ operators. For a given parallelepiped C^u and a $u-periodic$ $\mathbf{C}^D-valued$ function $\Psi(x), x \in \mathbf{Z}^d$, let us denote by $(\pi_{C^u}\Psi)(x), x \in C^u$ its restriction to C^u. Clearly π_{C^u} is a one-to-one correspondence between $u-periodic$ $\mathbf{C}^D-valued$ functions on $\mathbf{Z}^d$ and all $\mathbf{C}^D-valued$ functions on the parallelepiped C^u. The statement below is an immediate consequence of Lemma 2.2.

Corollary 2.7. *Suppose that* A *is a* $q-periodic$ *operator in* $\mathcal{L}_D$, $C \succeq C^q$ *and* $\Psi_C(x), x \in C$ *is a* $\mathbf{C}^D-valued$ *function on* C *then*

$$\overset{\circ}{A}_C\Psi_C = \pi_C A \pi_C^{-1}\Psi_C \tag{2.34}$$

In addition to that, if $\Psi(x), x \in \mathbf{Z}^d$, *is a* $u-periodic$ $\mathbf{C}^D-valued$ *function and* $C = C^u \succeq C^q$ *then*

$$A\Psi = \pi_C^{-1}\overset{\circ}{A}_C\pi_C\Psi \tag{2.35}$$

Lemma 2.8 . *Suppose that* A *is a* $q-periodic$ *operator in* $\mathcal{L}_D$ *and* $C^q \preceq C_1 \preceq C_2$. *Then the following is true*

$$\sigma(\overset{\circ}{A}_{C_1}) \subseteq \sigma(\overset{\circ}{A}_{C_2}) \tag{2.36}$$

Moreover, the eigenfunctions of the matrix $\overset{\circ}{A}_{C_1}$ *can be naturally extended to the corresponding eigenfunctions of the matrix* $\overset{\circ}{A}_{C_2}$.

Proof. To prove the inclusion suppose that λ is an eigenvalue of the matrix $\overset{\circ}{A}_{C_1}$. Then there is a function $\Psi_1(x), x \in C_1$ such that

$$\overset{\circ}{A}_{C_1}\Psi_1(x) = \lambda\Psi_1(x), x \in C_1 \tag{2.37}$$

Now, let us extend the function $\Psi_1(x)$ periodically on C_2 as follows

$$\Psi_2(x) = (\pi_{C_2}\pi_{C_1}^{-1}\Psi_1)(x), x \in C_2 \tag{2.38}$$

Then by a straightforward computation we obtain from (2.34) and $(\pi A\Psi)$ the following

$$\overset{\circ}{A}_{C_2}\Psi_2 = \pi_{C_2}A\pi_{C_1}^{-1}\Psi_1 = \pi_{C_2}\pi_{C_1}^{-1}\overset{\circ}{A}_{C_1}\Psi_1 = \lambda\Psi_2 \tag{2.39}$$

This means that $\lambda \in \sigma(\overset{\circ}{A}_{C_2})$ that completes the prove of the lemma. □

For the investigation of spectra we will need the following statement (e.g. [13]).

Lemma 2.9 . (distance to the spectrum). *Let $\mathcal{H}$ be a separable Hilbert space and A be a self-adjoint operator in $\mathcal{H}$. Then if $\sigma(A)$ is the spectrum of A and λ is a real number then*

$$\mathrm{dist}\{\sigma(A), \lambda\} = \min_{\Psi \in \mathcal{H}, \|\Psi\| = 1} \|(A - \lambda)\Psi\| \tag{2.40}$$

Proof of Theorem 4.

Let us prove first the inclusion in the formula (1.10). To do so assume that for a real λ there exist a natural n such that λ is an eigenvalue of the matrix $\overset{\circ}{A}_{C_n}$, i.e. $\lambda \in \sigma(\overset{\circ}{A}_{C_n})$ and there there is a vector $\Psi(x), x \in C_n$ such that

$$\overset{\circ}{A}_{C_n} \Psi(x) = \lambda \Psi(x), x \in C_n \tag{2.41}$$

Now, from (2.34) we easily obtain

$$(A \pi_{C_n}^{-1} \Psi)(x) = \lambda(\pi_{C_n}^{-1} \Psi)(x), x \in \mathbf{Z}^d, \tag{2.42}$$

which follows straightforwardly from the $q - periodicity$ of the operator A as an operator in $\mathcal{L}_D$. Then for any $m > n$ we define

$$\Psi_m(x) = (\pi_{C_n}^{-1} \Psi)(x), x \in C_m, \quad \Psi_m(x) = 0, x \notin C_m \tag{2.43}$$

Let us pick an arbitrary $\varepsilon > 0$ and introduce the following notation for a function $\Phi(x)$

$$\|\Phi(x)\|_{C_m}^2 = \sum_{x \in C_m} |\Phi(x)|^2 \tag{2.44}$$

Let us introduce also the for each $j, 1 \leq j \leq d, the$ number r_j which is the ratio of the corresponding edges of the parallelepipeds C_m and C_n. Then since the function $\pi_{C_n}^{-1} \Psi$ is periodic it is easy to see that

$$\|(A - \lambda)\Psi_m\|_{C_m}^2 \leq C(|C_m|/|C_n|)(\sum_{1 \leq j \leq d} r_j^{-1}) \|\Psi\|_{C_n}^2 \tag{2.45}$$

where $C = C(\rho, \|A\|)$ is a constant depending on the range ρ of localization for the operator A and its norm only (see Definitions in Section 1). From the definition of Ψ_m it follows that

$$\|\Psi_m\|^2 = \|\Psi_m\|_{C_m}^2 = (|C_m|/|C_n|) \|\Psi\|_{C_n}^2 \tag{2.46}$$

Besides, from the definition of the the sequence of parallelepipeds C_n it follows that for each j $r_j \to \infty$ when $m \to \infty$. If we set now $\widetilde{\Psi}_m(x) = \Psi_m(x)/\|\Psi_m(x)\|$ then from the relationships (2.45), (2.46) and the previous comment we obtain for any given $\varepsilon > 0$ and for sufficiently large m

$$\|(A-\lambda)\widetilde{\Psi}_m\| \le \varepsilon \tag{2.47}$$

¿From this and Lemma 2.9 we obtain the desired inclusion in (1.10). Therefore we have

$$\sigma(A) \supseteq \overline{\cup_{n\ge 1}\sigma(\mathring{A}_{C_n})} \tag{2.48}$$

To complete the proof we have to prove the inclusion opposite to the one above. If we pick again a positive ε then in view of Lemma 2.9 we can pick $\Psi \in l^2(\mathbf{Z}^d, \mathbf{C}^D)$ with norm 1 such that

$$\|(A-\lambda)\Psi\| \le \varepsilon \tag{2.49}$$

Now we define for any m

$$\Psi_m(x) = \Psi(x), x \in C_m, \quad \Psi_m(x) = 0, x \notin C_m \tag{2.50}$$

If $\widetilde{\Psi}_m(x) = \Psi_m(x)/\|\Psi_m(x)\|$ then since the operator A has a bounded norm and vector Ψ belong to the corresponding Hilbert space and has norm 1 we can pick a sufficiently large m such that

$$\|(A-\lambda)\widetilde{\Psi}_m\| \le 2\varepsilon \tag{2.51}$$

Now we note that for any $n > m$ by (2.35) we have

$$\pi_{C_n}(A-\lambda)\widetilde{\Psi}_m = (\mathring{A}_{C_n} - \lambda)\pi_{C_n}\widetilde{\Psi}_m \tag{2.52}$$

In addition to that, the definition of Ψ_m yields

$$\|\pi_{C_n}\widetilde{\Psi}_m\|_{C_n} = \|\widetilde{\Psi}_m\| = 1 \tag{2.53}$$

¿From (2.52), (2.53) and (2.51) we conclude that

$$\|(\mathring{A}_{C_n} - \lambda)\pi_{C_n}\widetilde{\Psi}_m\|_{C_n} \le 2\varepsilon \tag{2.54}$$

Therefore for any ε there is an n such that

$$\operatorname{dist}\{\sigma(\overset{\circ}{A}_{C_n}), \lambda\} \leq 2\varepsilon \tag{2.55}$$

¿From this we may conclude that

$$\sigma(A) \subseteq \overline{\cup_{n\geq 1}\sigma(\overset{\circ}{A}_{C_n})} \tag{2.56}$$

The last relationship together with (2.48) implies the equality in (1.10) that together with Lemma 2.8 completes the proof of Theorem 4. □

Lemma 2.10 . *Suppose that the operator* $A = B + \xi$ *acts in* $l_2(\mathbf{Z}^d)$ *where* B *is a* $q-$ *periodic self-adjoint operator and* $\xi(x)$ *is a* $u-$ *periodic real-valued function such that* $u \succeq q$ *and for some finite constants* $\xi_1, \xi_2 : \xi_1 \leq \xi(x) \leq \xi_2, x \in \mathbf{Z}^d$. *Then for any parallelepiped* $C \succeq C^u$ *the following is true*

$$\sigma(\overset{\circ}{A}_C) \subseteq \sigma(\overset{\circ}{B}_C) + [\xi_1, \xi_2] \subseteq \sigma(B) + [\xi_1, \xi_2] \tag{2.57}$$

$$\sigma(A) \subseteq \sigma(B) + [\xi_1, \xi_2] \tag{2.58}$$

Proof. Without loss of generality we may assume that $-\xi_1 = \xi_2 = \xi_0$ where ξ_0 is a nonnegative constant since we can always redefine A as $A = (B+t)+(\xi-t), t = (\xi_2-\xi_1)/2$. Keeping this in mind let us note now that for any two linear bounded operators D_1 and D_2

$$\sigma(D_1) \subseteq \sigma(D_2) + [-d, d], d = \|D_1 - D_2\| \tag{2.59}$$

Indeed, if $\lambda \notin \sigma(D_2) + [-d, d]$ then $\|(D_2 - \lambda)^{-1}\| < d^{-1}$ and therefore $(D_1 - \lambda)^{-1}$ is clearly a bounded operator that implies (2.59). Since $\|\xi\| \leq \xi_0$ then (2.59) implies the first inclusion in (2.57) and (2.58). The second inclusion in (2.57) follows from the first one and (1.10). The lemma is therefore proved. □

Proof of Theorem 2.

Let us note that without loss of generality we may assume that $u = (u_1, \ldots, u_d)$ and the parameter ρ associated with a $u-$*periodic* local operator A satisfy the following inequality

$$\min_{1 \le j \le d} u_j > 2\rho + 1 \tag{2.60}$$

If not we may always pick $u' \succ u$ such that u' satisfy (2.60) and treat A as $u' - periodic$. We shall assume from now on that the inequality (2.60) is satisfied for any period u we consider, in particular for $u = q$.

We have defined the periodic restriction $\overset{\circ}{A}_C$ for any $q - periodic$ operator for $C = C^u, u \succeq q$. We need to extend properly this definition for local operators A which are not necessarily periodic. This can be done as follows. First of all given a parallelepiped $C = C^u + l$ we construct an appropriate $u - periodic$ operator associated with C and A which we shall denote by $A^{(C)}$. We note that for a local operator A the representation (2.2) is clearly still valid. We want to preserve the self-adjointness for $A^{(C)}$ if A is self-adjoint. The operator A is self-adjoint if and only if the constraints (2.3) hold. In order to provide these constraints we represent the set $\{z \in \mathbf{Z}^d : |z| \le \rho\} = \{0\} \cup Z \cup (-Z)$ in such a way that $0 \notin Z \cup (-Z)$ and $Z \cap (-Z) = \emptyset$. Clearly we can always do this. Then we may set $a_z, z \in Z \cup \{0\}$ as we wish and define $a_z, z \in (-Z)$ by the equalities (2.3). Now we define a linear operator τ_C which maps any $\mathbf{C}^D - valued$ function $a(x), x \in \mathbf{Z}^d$, onto a $u - periodic$ function $\tau_C a$ as follows

$$a_z^{(C)}(x) = \tau_C a(x) = a(x), x \in C, a_z^{(C)}(x + un) = a_z^{(C)}(x), x \in \mathbf{Z}^d \tag{2.61}$$

In other words, $\tau_C a$ is a $u - periodic$ extension of a coinciding with the function a on the parallelepiped $C = C^u + l$. Now since A is represented by (2.2) we define an associated $u - periodic$ operator $A^{(C)}$ by the same formula (2.2) where the $a_z, z \in Z \cup \{0\}$ are replaced by $a_z^{(C)}, z \in Z \cup \{0\}$ and the remaining functions $a_z^{(C)}, z \in (-Z)$ are defined to keep the constraints (2.3). With this definition the $u - periodic$ operator $A^{(C)}$ associated with the self-adjoint operator A and the parallelepiped $C = C^u + l$ is also self-adjoint. Having this we define the periodic restriction $\overset{\circ}{A}_C$ of a local operator A on a parallelepiped $C = C^u + l$ using (1.8) as follows

$$\overset{\circ}{A}_C = [\overset{\circ}{A}{}^{(C)}]_{C^u}, C = C^u + l \tag{2.62}$$

Definition. *We say that a point x is a boundary point of a parallelepiped C if there exists j, $1 \le j \le d$ such that either $x + e_j \notin C$ or $x - e_j \notin C$. The set of boundary points is denoted by ∂C.*

The statement below shows that the periodic restriction of A on C does not differ much from the regular restriction $A(x, y), x, y \in \mathbf{Z}^d$.

Lemma 2.11 . *Let A be a local operator. If $C = C^u + l$, $l \in \mathbf{Z}^d$ then the following equalities are true*

$$\overset{\circ}{A}_C(x,y) = A(x,y), x,y \in C, dist\{x, \partial C\}, dist\{y, \partial C\} > \rho \tag{2.63}$$

where $\operatorname{dist}\{x, \partial C\} = \max_{z \in \partial C} |x - z|_\infty$. *If A is a self-adjoint operator then $\overset{\circ}{A}_C$ is self-adjoint as well.*

Proof. The statements of the lemma follows straightforwardly from (1.8), (2.61), (2.62) and (2.60). □

The construction of the periodic restrictions is clearly applicable to the operators $H = H_0 + gv$ defined by (1). Whenever we shall need to emphasize that H depends on v we write $H = H(v)$.

Lemma 2.12 . *The spectrum of the operator H is nonrandom with probability* 1, *i.e. there exists a closed set $\sigma \subseteq \mathbf{R}$ such that with probability* 1 $\sigma(H) = \sigma$.

Proof. We note that the operator H is metrically transitive and then we can just reference to [14]. □

Let $\mathcal{P}_q$ be the set of real-valued functions $\xi(x)$ which are $u-periodic$ for some $u \succeq q$ and satisfy $\xi_1 \le \xi(x) \le \xi_2$.

Theorem 2.13 . *Suppose that $C_n, n = 1, 2, \ldots$ is a sequence of parallelepipeds such that $C^q \preceq C_n \prec C_{n+1}, n \ge 1$. Let the operator $H = H_0 + \xi$ and the spectrum σ be defined as in Theorem 2. Then the nonrandom spectrum σ of the operator H can be represented as follows*

$$\sigma = \overline{\cup_{\xi \in \mathcal{P}_q} \sigma[H(\xi)]} = \overline{\cup_{n \ge 1, \xi \in \mathcal{P}_q} \sigma[\overset{\circ}{H}_{C_n}(\xi)]} = \sigma(\xi_1, \xi_2) \tag{2.64}$$

where

$$\sigma(\xi_1, \xi_2) = \sigma(H_0) + [\xi_1, \xi_2] \tag{2.65}$$

Proof. First of all we note that the following equalities are true

$$\overline{\cup_{\xi \in \mathcal{P}_q} \sigma[H(\xi)]} = \overline{\cup_{n \ge 1, \xi \in \mathcal{P}_q} \sigma[\overset{\circ}{H}_{C_n}(\xi)]} = \sigma(H_0) + [\xi_1, \xi_2] \tag{2.66}$$

These inequalities follow straightforwardly from Theorem 4 and Lemmas 2.10 if we note that for a $u - periodic$ ξ from $\mathcal{P}_q$ the operator $H(\xi)$ is $u - periodic$ and, in addition to that, we may set $\xi(x) \equiv t, where\ t$ is a constant such that $-1 \le t \le 1$.

Recall now that the function $\xi(x)$ is a random function, i.e. we have a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $\xi(x) = \xi_\omega(x)$ where ω is a realization from Ω. Let us observe that as it follows from Lemma 2.12 there exist a set $\Omega_1 \subseteq \Omega$ such that $\mathbf{P}(\Omega_1) = 1$ and

$$\sigma(H(\xi_\omega)) = \sigma, \omega \in \Omega_1 \tag{2.67}$$

Let us pick any positive ε and ω such that (2.67) is true. Assume that $\lambda \in \sigma$. Then in view of Lemma 2.9 there exists m and a vector Ψ in the Hilbert space such that $||\Psi|| = 1$ and

$$||(H(\xi_\omega) - \lambda)\Psi|| \le \varepsilon, \Psi(x) = 0, x \notin C_m. \tag{2.68}$$

We may impose the extra constraint $\Psi(x) = 0, x \notin C_m$ on the vector Ψ since the operator H is local and bounded. Then for any $n > m$

$$H(\xi_\omega)\Psi(x) = \overset{\circ}{H}_{C_n}(\xi_\omega)\Psi(x), x \in C_n \tag{2.69}$$

and, therefore,

$$||(\overset{\circ}{H}_{C_n}(\xi_\omega) - \lambda)\Psi||_{C_n} \le \varepsilon \tag{2.70}$$

The last equality implies that

$$\lambda \in \overline{\cup_{n\ge1,\xi\in\mathcal{P}_q}\sigma[\overset{\circ}{H}_{C_n}(\xi)]}$$

and consequently

$$\sigma \subseteq \overline{\cup_{n\ge1,\xi\in\mathcal{P}_q}\sigma[\overset{\circ}{H}_{C_n}(\xi)]} \tag{2.71}$$

To prove the opposite inclusion, let us pick again a positive ε and a $u - periodic$ $\xi \in \mathcal{P}_q$. Then we suppose that $\lambda \in \sigma[H(\xi)]$. Since the operator H is local and bounded we can apply again Lemma 2.9 and get for a natural m the equality (2.68) with ω dropped , i.e. there exists a vector Ψ, $||\Psi|| = 1$ such that

$$||(H(\xi) - \lambda)\Psi|| \le \varepsilon, \Psi(x) = 0, x \notin C_m \tag{2.72}$$

Now we note that in view of the conditions imposed on $\xi_\omega(x)$(*see* Theorem 2) for any positive δ there exist a set Ω_ξ, $\mathbf{P}(\Omega_\xi) = 1$ such that

$$\forall \delta, \forall \omega \in \Omega_\xi : \exists l = l(\delta,\omega) \in \mathbf{Z}_u^d : \quad \max_{x \in C_m + l} |\xi_\omega(x) - \xi(x)\| \le \delta \tag{2.73}$$

Moreover, if we denote $\Psi_l(x) = \Psi(x-l)$ then since ξ is $u-periodic$ we have from (2.72)

$$\forall l \in \mathbf{Z}_u^d : \|(H(\xi) - \lambda)\Psi_l\| \le \varepsilon, \tag{2.74}$$

Clearly, if we pick δ small enough then

$$\forall \omega \in \Omega_\xi : \exists l = l(\varepsilon,\omega) \in \mathbf{Z}_u^d : \|(H(\xi_\omega) - \lambda)\Psi_l\| \le 2\varepsilon, \tag{2.75}$$

¿From this we immediately obtain

$$\sigma \supseteq \sigma[H(\xi)], \xi \in \mathcal{P}_q \tag{2.76}$$

and consequently

$$\sigma \supseteq \overline{\quad \cup_{\xi \in \mathcal{P}_q} \sigma[H(\xi)]} \tag{2.77}$$

Thus, (2.66), (2.71) and (2.77) imply the desired relationships (2.64) that completes the prove of the theorem. □

In order to use the multiscale analysis [9] we need to get exponential estimates for the resolvent of the operators H and their periodic restrictions. For this purpose we will adapt the Combes-Thomas argument to our operators. We start with a description of the relevant resolvents. Let us denote by $b_x, x \in \mathbf{Z}^d$, the standard basis in the the space $l^2(\mathbf{Z}^d)$, i.e. $b_x(x) = 1, b_x(y) = 0, y \ne x, y \in \mathbf{Z}^d$. In the case of $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ we introduce the basis $b_{\alpha,x}, \alpha = 1,\ldots,d$, i.e., $b_{\alpha,x}(\alpha,x) = 1$, and $b_{\alpha,x}(\beta,y) = 1$, if $\beta \ne \alpha$ or $y \ne x$, $\beta = 1,\ldots,d, y \in \mathbf{Z}^d$. Supposing that A is a local operator (not necessarily periodic) acting in $l^2(\mathbf{Z}^d)$ or in $l^2(\mathbf{Z}^d, \mathbf{C}^D)$ with entries $A(x,y), x,y \in \mathbf{Z}^d$. For such an operator the representation (2.2) is still applicable. Then if ζ is a complex or real number and $\zeta \notin \sigma(A)$, we may consider for the cases $l^2(\mathbf{Z}^d)$ or $l^2(\mathbf{Z}^d, \mathbf{C}^D)$, respectively, the Green's functions

$$G(\zeta,x,y) = (b_x, (H-\zeta)^{-1} b_y), x,y \in \mathbf{Z}^d \tag{2.78}$$

$$G(\zeta,x,y) = G(\zeta,\alpha,x,\beta,y) = (b_{\alpha,x}, (H-\zeta)^{-1} b_{\beta,y}), \alpha,\beta = 1,\ldots,d,\ x,y \in \mathbf{Z}^d \tag{2.79}$$

We will often drop α and β in the notation of the resolvent for briefness.

Lemma 2.14 . *Suppose that A is a local operator described above such that for a positive constant c we have $|A(x,y)| \leq c, x, y \in \mathbf{Z}^d$. Suppose also that that*

$$\operatorname{dist}\{\zeta, \sigma(A)\} = \delta > 0 \tag{2.80}$$

Then there exists a positive constant $b = b(c, \rho)$ (ρ is the number associated with the local operator A) such that

$$|G(\zeta, x, y)| \leq 2\delta^{-1} e^{-b\delta|x-y|}, x, y \in \mathbf{Z}^d \tag{2.81}$$

where

$$|x| = \sum_{1 \leq j \leq d} |x_j| \tag{2.82}$$

Moreover, if A is $u-$periodic operator then the following identity is true

$$G(\zeta, x+u, y+u) = G(\zeta, x, y), x, y \in \mathbf{Z}^d \tag{2.83}$$

Proof. For $\alpha \in \mathbf{C}^d$ let M_α be the operator given by multiplication by

$$M_\alpha(x) = e^{2\pi i(\alpha, x)}, x \in \mathbf{Z}^d \tag{2.84}$$

Then in view of (2.2) and (2.4) we have

$$A(\alpha) = M_\alpha A M_\alpha^{-1} = \sum_{|z| \leq \rho} a_z V(\alpha)^z, V_j(\alpha) = e^{2\pi i \alpha_j} V_j, 1 \leq j \leq d \tag{2.85}$$

Note that $A(\alpha)$ coincides with the relevant operator in (2.29) but now $\alpha \in \mathbf{C}^d$. Clearly, the last representation implies the existence of a constant $K = K(c, \rho)$ such that

$$\|A - A(\alpha)\| \leq K|\alpha| \tag{2.86}$$

In view of (2.80) we have immediately $\|G(\zeta)\| \leq \delta^{-1}$. This inequality together with the inequality (2.86) imply for $G(\alpha, \zeta) = (A(\alpha) - \zeta)^{-1}$

$$\|G(\alpha, \zeta)\| \leq 2\delta^{-1}, |\alpha| < \delta/(2K) \tag{2.87}$$

Now we note that

$$[G(\alpha, \zeta)](x, y) = G(\zeta, x, y) exp\{2\pi i \alpha \cdot (x - y)\}, x, y \in \mathbf{Z}^d \tag{2.88}$$

¿From this and the obvious inequality $|G(\alpha, \zeta)](x, y)| \leq \|G(\alpha, \zeta)\|$ we obtain the inequality (2.81) by taking an appropriate α .

The identity (2.83) is a direct consequence of the $u-$*periodicity* of the operator A. This completes the proof of the lemma. □

Lemma 2.15 . *Suppose that the conditions of Lemma 2.14 are satisfied and let us consider for* $C = C^u + l, l \in \mathbf{Z}^d$ *the resolvent*

$$G'_C(\zeta, x, y) = [(\overset{\circ}{A}_C - \zeta)^{-1}](x, y), x, y \in C \tag{2.89}$$

Then the following estimate is true

$$|G'_C(\zeta, x, y)| \leq 2\delta^{-1}(1 + 2\Pi(v, \delta))e^{-b\delta|x-y|_u}, x, y \in C \tag{2.90}$$

where b *is the same constant as in Lemma 2.14 and*

$$\Pi(v, \delta) = \prod_{1 \leq j \leq d} \left(1 - e^{-b\delta|u_j|}\right)^{-1}, \ |x - y|_u = \min_{n \in \mathbf{Z}^d} |x - y - nu| \tag{2.91}$$

Proof. We note first that in view of the definition of the periodic restriction $\overset{\circ}{A}_C$ (2.62) we may assume without loss of generality that A is a $u-$*periodic* operator and $C = C^u$. Keeping this in mind and using (2.83) together with the following identity

$$\sum_{y \in \mathbf{Z}^d} (A(x, y) - \zeta)G(\zeta, y, z) = \delta_{x,z}, x, z \in \mathbf{Z}^d \tag{2.92}$$

where $\delta_{x,z}$ is the delta-function we obtain

$$\sum_{n \in \mathbf{Z}^d} \sum_{y \in \mathbf{Z}^d} (A(x, y) - \zeta)G(\zeta, y, z + un) = \sum_{n \in \mathbf{Z}^d} \delta_{x,z+un}, x, z \in C \tag{2.93}$$

¿From this, (1.8) and (2.5) we obtain

$$\sum_{y \in C} (\overset{\circ}{A}_C(x, y) - \zeta)\overset{\circ}{G}_C(\zeta, y, z) = \delta_{x,z}, x, z \in C \tag{2.94}$$

Therefore,

$$G'_C(\zeta, x, y) = \overset{\circ}{G}_C(\zeta, x, y) = \sum_{n \in \mathbf{Z}^d} G(\zeta, x, y + un), x, y \in C \tag{2.95}$$

¿From this and the previous lemma we immediately obtain

$$|G'_C(\zeta, x, y)| \leq 2\delta^{-1} \sum_{n \in \mathbf{Z}^d} e^{-b\delta|x-y-nu|}, x, y \in C \tag{2.96}$$

If we recall the definition (2.91) of $|x - y|_u$ one can easily prove that there is $n' \in \mathbf{Z}^d$ such that

$$|x-y|_u = |z|, z = x-y-n'u = cu, 0 \le |c_j| \le 1/2, 1 \le j \le d \tag{2.97}$$

Now we rewrite the right side of the inequality (2.96) using (2.82) as follows

$$\sum_{n\in\mathbf{Z}^d} e^{-b\delta|x-y-nu|} = \sum_{n\in\mathbf{Z}^d} e^{-b\delta|cu-nu|} = \prod_{1\le j\le d} \sum_{n\in\mathbf{Z}} e^{-b\delta|c_j-n||u_j|} \tag{2.98}$$

We shall need the following elementary inequality

$$\sum_{n\in\mathbf{Z}} e^{-c|m-n|} \le e^{-c|m|}\left[(1+2(1-e^{-c})^{-1}\right], 0 \le |m| \le 1/2, c > 0 \tag{2.99}$$

which can be verified by a direct computation. Applying this inequality to the right side of (2.98) and combining the result with the inequality (2.96) we get the desired estimate (2.90). The lemma is proved. □

Proof of Theorem 3.

Let us consider the left edge λ_i of the gap (λ_i, μ_i), the right edge μ_i can be treated in a similar way. We will use the conditions for localization given in Theorem 2.1 of von Dreifus and Klein [9]. We start with some definitions. For $u \in \mathbf{Z}^d$ we define $H^{(u)}$ by

$$H_0^{(u)}(x,y) = H_0(x+u, y+u), x, y \in \mathbf{Z}^d \tag{2.100}$$

We then set

$$H^{(u)} = H_0^{(u)} + gv, G^{(u)}(\zeta) = (H_0^{(u)} - \zeta)^{-1} \tag{2.101}$$

Notice that $\sigma(H^{(u)}) = \sigma$ with probability 1. For $l \in N, x \in \mathbf{Z}^d$, we define $\widetilde{l} = l(1,\ldots,1)$ and $\Lambda_l(x) = C^{\widetilde{l}} - [l/2] + x$ ($[y]$ is the entire part of a real number y) and for $\Lambda \subset \mathbf{Z}^d$

$$\partial_\rho \Lambda = \{y \in \Lambda : \exists z \in \mathbf{Z}^d - \Lambda, |z-y|_\infty \le \rho\} \tag{2.102}$$

Recall that ρ is the range of H_0. Also for $\Lambda \in \mathbf{Z}^d$ we write $H_\Lambda = \{H(x,y), x, y \in \Lambda\}$ which is the matrix associated with the restriction of H to Λ with Dirichlet boundary conditions.

Definition 2.16. *Let $x \in \mathbf{Z}^d$, $E \in \mathbf{R}$, $m > 0$, $l > \rho$. We say that $\Lambda_l(x)$ is $(m,E)-regular$ if*

$$\max_{u \in C^q} |G^{(u)}_{\Lambda_l(x)}(E;x,y)| \le e^{-ml/2}, \forall y \in \partial_\rho \Lambda_l(x) \tag{2.103}$$

Otherwise we say that $\Lambda_l(x)$ *is* $(m, E) - singular.$

Let us fix $p > d$, an interval $I \subset \mathbf{R}$, m_0 and D_0 (see Assumption V). The von Dreifus-Klein criterion says that there exists $B = B(d, D_0, m_0, p) < \infty$ such that if

$$\mathbf{P}\{\Lambda_{L_0}(x) \text{ is } (m_0, E) - regular \text{ for all } E \in I\} \geq 1 - \frac{1}{L_0^p} \tag{2.104}$$

for some $L_0 > B$, then there exists $\delta = \delta(L_0, m_0, d, D_0, p) > 0$ such that the spectrum of H is exponentially localized in $(E_0 - \delta, E_0 + \delta)$.

Remark 2.17 . *von Dreifus and Klein only discuss the case where* $H = -\Delta + gv$. *But their results are easily seen to extend to the case when* $-\Delta$ *is replaced by a translation invariant operator with a finite range* ρ. *The remark that* $-\Delta$ *can be replaced by a* $q-$ *periodic operator* H_0 *is due to Spencer [16], who noticed that if the maximum over all translations of* H_0 *is introduced in the definition* (2.103), *the whole proof goes through.*

Theorem 3 now follows from

Lemma 2.18 . *Let us fix* $0 < \Omega_+ < 1$, *and let* $p_+ = \mu\{[\Omega_+, 1]\}$, $g_+ = g(1 - \Omega_+)$. *If* L *is a sufficiently large positive integer such that* $\widetilde{L} \succeq q$, *we have*

$$\lim_{p_+ \to 0} \mathbf{P}\{\Lambda_L(0) \text{ is } (b(g_+ - g')/4, \lambda) - regular\} = 1 \tag{2.105}$$

uniformly in $\lambda \in [\lambda_i - g', \lambda_i]$ *for* g', $0 < g' < g_+$, *where* b *is given in Lemma 2.14.*

Proof. Let $\mathcal{E}_L$ denote the event that $v(x) \leq \Omega_+$ for for all $x \in \Lambda_L(0)$. If $\mathcal{E}_L$ occurs, and $0 < g' < g_+$, then for all $u \in C^q$ we have from (2.90) that for all $\lambda \in [\lambda_i - g', \lambda_i]$

$$|\overset{\circ}{G}{}^{(u)}_{\Lambda_L(0)}(\lambda; x, y)| \leq \frac{2^{d+1}}{g_+} exp(-bg''|x-y|_{\widetilde{L}}) \tag{2.106}$$

for L sufficiently large in relation to q, for all $x, y \in \Lambda_L(0)$, where $g'' = g_+ - g'$. Define now $\Gamma_L^{(u)}$ by the following equality

$$H^{(u)}_{0,\Lambda_L(0)} = \overset{\circ}{H}{}^{(u)}_{0,\Lambda_L(0)} + \Gamma_L^{(u)} \tag{2.107}$$

i.e. $\Gamma_L^{(u)}$ is the difference between matrices corresponding to the periodic and Dirichlet boundary conditions. Notice that $\|\Gamma_L^{(u)}\| \leq C(H_0)$, where $C(H_0)$ is a constant which depends just on operator H_0. Then if G_Λ stands for the resolvent of the corresponding matrix H_Λ, the resolvent identity gives

$$G^{(u)}_{\Lambda_L(0)}(\lambda) = \overset{\circ}{G}{}^{(u)}_{\Lambda_L(0)}(\lambda) + \overset{\circ}{G}{}^{(u)}_{\Lambda_L(0)}(\lambda)\Gamma^{(u)}_L G^{(u)}_{\Lambda_L(0)}(\lambda) \tag{2.108}$$

$$G^{(u)}_{\Lambda_L(0)}(\lambda;0,y) = \overset{\circ}{G}{}^{(u)}_{\Lambda_L(0)}(\lambda;0,y) + \sum_{s,t\in\Lambda_L(0)} \overset{\circ}{G}{}^{(u)}_{\Lambda_L(0)}(\lambda;0,t)\Gamma^{(u)}_L(t,s)G^{(u)}_{\Lambda_L(0)}(\lambda;s,y)$$

If $y \in \partial_\rho\Lambda_L(0)$, then using (2.106) we get

$$|G^{(u)}_{\Lambda_L(0)}(\lambda;0,y)| \leq \tag{2.109}$$

$$\leq \frac{2^{d+1}}{g''}e^{-bg''(\frac{L}{2}-\rho)} + (2L+1)^{2d}C(H_0)\|G^{(u)}_{\Lambda_L(0)}(\lambda)\|e^{-bg''(\frac{L}{2}-\rho)}$$

since $\Gamma^{(u)}_L(t,s) = 0$ unless $s,t \in \partial_\rho\Lambda_L(0)$. Now let $\mathcal{W}_L(\lambda)$ be the event $\|G^{(u)}_{\Lambda_L(0)}(\lambda)\| \leq L^{2d}$, for all $u \in C^q$. Then we get

$$|G^{(u)}_{\Lambda_L(0)}(\lambda;0,y)| \leq \frac{2^{d+1}}{g''}exp\{-bg''(\tfrac{L}{2}-\rho)\}[1+(2L+1)^{2d}C(H_0)L^{2d}] \leq \tag{2.110}$$

$$\leq exp\{-\frac{bg''L}{8}\}$$

for all $\lambda \in [\lambda_i - g', \lambda_i]$, if L is greater than a finite constant $L'(d,b,g'',H_0)$. Thus

$$\mathbf{P}\{\Lambda_L(0) \text{ is } (\tfrac{bg''}{4},\lambda) - singular\} \leq \mathbf{P}\{\mathcal{E}^c_L\} + \mathbf{P}\{\mathcal{W}^c_L(\lambda)\} \tag{2.111}$$

On the other hand, for all $\lambda \in [\lambda_i - g', \lambda_i]$

$$\mathbf{P}\{\mathcal{E}^c_L\} \leq L^d\mathbf{P}(v(0) > \Omega_+\} \leq p_+L^d \tag{2.112}$$

and by Wegner's estimate

$$\mathbf{P}\{\mathcal{W}_L(\lambda)\} \leq \frac{2D_0}{g}|C^q|\frac{L^d}{L^{2d}} = \frac{2D_0}{g}|C^q|L^{-d} \tag{2.113}$$

This completes the proof of the lemma, and hence Theorem 3. □

Proof of Theorem $3'$. We use the localization criterion given by Spencer [15]. The proof is similar to the proof of Theorem 3, so we will only point out the differences. Lemma 2.18 is replaced by

Lemma 2.19 . *Let $m_L = 2(d+2)logL/L$. Under the hypotheses of Theorem 3′ we have*

$$\limsup_{L \to \infty} \mathbf{P}\{\Lambda_L(0) \text{ is } (m_L, \lambda_i) - regular\} = 1 \tag{2.114}$$

Proof. The lemma is proved in a similar way to Lemma 2.18, for scales such that $\widetilde{L} \succeq q$. Here we define $\mathcal{E}_L$ to be the event that $v(x) \leq 1 - \delta_L$ for all $x \in \Lambda_L(0)$, where $\delta_L = (logL)^2/L$. By our assumptions we have

$$\mathbf{P}\{\mathcal{E}_L^c\} \leq L^d \mathbf{P}(v(0) > 1 - \delta_L\} \leq CL^d \delta_L^\eta = CL^d \frac{(logL)^{2\eta}}{L^\eta} \to 0 \text{ as } L \to \infty \tag{2.115}$$

since $\eta > d$. ∎

Theorem 3′ now follows from Theorem 1 in [15].

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